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[3] Probability Theory

0. Outline

1. **Probability theory basics (ctd.)**
2. **Countable additivity**

1. Probability theory basics (ctd.)

- Let Ω denote a non-empty set (sometimes known as the *sample space*) and Ø denote the empty set.
- A *σ-field* (‘sigma-field’ aka ‘sigma-algebra’) $\mathcal{F}$ of subsets of Ω is a set of subsets of Ω, containing Ω and closed under countable intersection and complementation (and hence countable union).

  E.g.: {Ø, Ω} is a σ-field, as Ω is the complement of Ø and $\Omega \cap \text{Ø} = \text{Ø}$. $\wp(\Omega)$ is also obviously a σ-field.
- In other words, it obeys the following constraints:

  [σ1] $\Omega \in \mathcal{F}$

  [σ2] If $P \in \mathcal{F}$, then $\bar{P} \in \mathcal{F}$

  [σ3] For any countable set of pairwise disjoint propositions $\{P_1, P_2, \ldots\}$, if $P_1, P_2, \ldots \in \mathcal{F}$, then $\bigcap P_i \in \mathcal{F}$.

1. Probability theory basics (ctd.)

- A *probability function* Pr is certain kind of function assigning a real-valued number to each member of a σ-field $\mathcal{F}$ defined on Ω.

  (Note: the domain of $\text{Bel}_{S,t}$ will be a set of *propositions* (subsets of the set of all possible worlds) and according to PROB, if *S* is rational at *t*, then this set will be a *field* of propositions).
- The ordered triple $\mathcal{M} = \langle \Omega, \mathcal{F}, \Pr \rangle$ is known as a *probability model* (or probability space).
- If Pr is a (Kolmogorov) *finitely additive* probability function, it obeys the following constraints for all *P* and *Q* belonging to $\mathcal{F}$:

  [P1] *Non-negativity:* $0 \leq \Pr(P)$

  [P2] *Unit normalisation:* $\Pr(\Omega) = 1$

  [P3] *Finite additivity:* If $P \cap Q = \text{Ø}$ then $\Pr(P \cup Q) = \Pr(P) + \Pr(Q)$.

1. Probability theory basics (ctd.)

- It is easy to prove that [P3] generalises to the *finite* union of more than two propositions, i.e.:

  [T1] For any finite set of pairwise disjoint propositions in $\mathcal{F}$ $\{A_1,\ldots, A_n\}$, the probability of the union of these propositions is equal to the sum of their individual probabilities (in formal notation: $\Pr(\bigcup A_i) = \sum \Pr(A_i)$).
- Note: whilst our presentation of probability theory has been couched in set theoretic terms, probabilities can also equivalently be defined over a set of *sentences of a formal language* $\mathcal{L}$, closed under union and negation.
- This is commonplace in the philosophical (as opposed to mathematical) literature.

1. Probability theory basics (ctd)

- Our three axioms [P1]-[P3] are simply translated as follows, for all sentences P and Q belonging to $\mathcal{L}$:

  [P1'] $0 \leq \Pr(P)$

  [P2'] if $\vDash_{\mathcal{L}} P$, then $\Pr(P) = 1$

  [P3'] if $\vDash_{\mathcal{L}} \neg(P\&Q)$, then $\Pr(P \vee Q) = \Pr(P) + \Pr(Q)$
- The difference isn't entirely cosmetic.
- On the approach we have taken so far:

  Believing that X is trilateral = believing that X's internal angles sum to 180°, as $\{w \mid X \text{ is trilateral in } w\} = \{w \mid X\text{'s internal angles sum to } 180° \text{ in } w\}$.

1. Probability theory basics (ctd)

- On the 'linguistic' approach, we can deny this:

  Believing that X is a triangle ≠ believing that X's internal angles sum to 180°, as the former involves a relation to the atomic sentence 'X is a triangle' and the latter a relation to the different atomic sentence 'X's internal angles sum to 180°'
- For further discussion of some of these issues, see Chalmers [ms].
- From [P1]-[P3], one can derive a number of useful theorems :

  [T2] $\Pr(P) = 1 - \Pr(\overline{P})$

  Proof: $P \cup \overline{P} = \Omega$, hence (by [P2]) $\Pr(P \cup \overline{P}) = 1$. Furthermore, $P \cap \overline{P} = \varnothing$, hence (by [P3]) $\Pr(P) + \Pr(\overline{P}) = 1$, from which [T2] immediately follows. ■

1. Probability theory basics (ctd)

[T3] *Zero normalisation:* $\Pr(\varnothing) = 0$

Proof: $\overline{\varnothing} = \Omega$, hence (by [T2]&[P2]) $\Pr(\varnothing) = 1 - \Pr(\Omega) = 0$. ■

[T4] $0 \leq \Pr(P) \leq 1$

[T5] $\Pr(P \cup Q) = \Pr(P) + \Pr(Q) - \Pr(P \cap Q)$

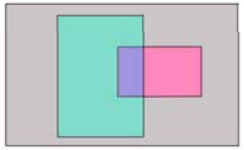
[T6] *Monotonicity:* If $P \subseteq Q$, then $\Pr(P) \leq \Pr(Q)$

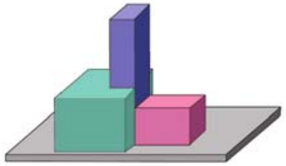
[T7] *Total Probability:* $\Pr(P) = \Pr(P \cap Q) + \Pr(P \cap \overline{Q})$

(I'll leave the proof of the last 4 theorems as an exercise)

1. Probability theory basics (ctd)

- Probability models can be visualised with:
  - A standard Venn diagram in which the probabilities of events are either (i) represented by the ratio of their area to the total area of the diagram or (ii) simply written onto the diagram.
  - A 3D (aka 'muddy') Venn diagram, in which the probabilities of events are represented by relative volume.





1. Probability theory basics (ctd)

- The previous theorems can easily be verified visually on these kinds of representations. Take some time outside class to have a go at this.

2. Countable additivity

- In the previous slides, we gave a definition of a *finitely additive* probability function, which guarantees that:

  The probability of the union of a *finite* number of disjoint propositions is equal to the sum of their individual probabilities.
- Note that *nothing* is guaranteed here with respect to the probability of the union of an *infinite* number of disjoint propositions.
- There does exist a special class of probability function, which makes additional requirements of this sort… at least with respect to the probability of the union of a *countably infinite* number of disjoint propositions.
- Reminder: a set is countably infinite iff its cardinality is = to the cardinality of the natural numbers. (see L1)

2. Countable additivity

- The narrower definition of a *countably additive* probability function is obtained by replacing [P3] with:

  [P3*] *Countable additivity:* for any countable set of pairwise disjoint propositions in $\mathcal{F}$ $\{A_1,\ldots, A_n\}$, the probability of the union of these propositions is equal to the sum of their individual probabilities (in formal notation: $\Pr(\bigcup A_i) = \sum \Pr(A_i)$).
- Let me explain exactly *why* this is a narrower definition ([P1] & [P2] & [P3*] obviously entails [P1] & [P2] & [P3], as a set is countable iff it is either finite or denumerable; let's demonstrate that the entailment doesn't go the other way)…
- Consider the set S of all the finite subsets of $\mathbb{N}$ and their complements (e.g. Ø and $\mathbb{N}$, $\{0\}$ and $\{1, 2, 3, \ldots\}$, etc.).

2. Countable additivity

- Consider the function f: $S \mapsto [0,1]$ such that f(A) = 0 if A is finite and f(A) = 1 otherwise.
- f satisfies [P1] as either f(P) = 0 or f(P) = 1 and f satisfies [P2] as f(ℕ) = 1, since ℕ is infinite.
- f satisfies [P3]:
  - Any two disjoint $X, Y \in S$ are such that either X and Y are both finite (case 1) or one is finite and the other not (case 2). They can't both be infinite as they wouldn't be disjoint.
  - Case 1: f($X \cup Y$) = 0 (the union is finite) and hence f($X \cup Y$) = 0 + 0 = f(X) + f(Y)
  - Case 2: f($X \cup Y$) = 1 (the union is infinite) and hence f($X \cup Y$) = 1 + 0 = f(X) + f(Y)

2. Countable additivity

  - f *fails* to satisfy [P3*]. Consider the union of all disjoint sets $\{i\}$ with $i \in$ ℕ (this union is simply ℕ itself). f($\{i\}$) = 0, for each $\{i\}$, as each of these sets is finite. But f(ℕ) = 1. ■
- Now, whilst [P3] might seem like a natural extension of [P3*], there is in fact a certain amount of controversy regarding whether if S is rational at t, $\text{Bel}_{S,t}$ should required to be a countably additive probability function or merely a finitely additive one.
- De Finetti's *guess-the-number problem* (De Finetti [1974]). Paul has to guess exactly which natural number John is thinking of.

  [1] It seems rationally permissible for Paul to hold an equal degree of belief = d in each of the members of the countably infinite set $\{A_1, A_2, \ldots\}$ (where A_i = the set of p-worlds in which John is thinking of natural number i).

2. Countable additivity

[2] If [P1], [P2] and [P3*] are rational requirements on degrees of belief, then Paul *isn't* rationally entitled hold an equal degree of belief = d in each of the aforementioned alternatives.

Proof: by [P2], Paul's degree of belief in the union of the possibilities must be equal to 1. Now by [P1], d cannot be strictly negative. So either $d = 0$ or $d > 0$. By [P3*]: (i) if $d = 0$, then Paul's degree of belief in the union of the possibilities = 0 (contradiction), (ii) if $d > 0$, Paul's degree of belief in the union of the possibilities = $+\infty$ (contradiction). ■

To be continued next time…

Reference

- Chalmers, D. [ms]: 'Probability and Propositions'. Available at consc.net/papers/probability.pdf
- De Finetti, B. [1974]: *Theory of Probability.* John Wiley, New York.

Next lecture: ‘More Probability Theory’

- Optional reading (*slightly* technical but accessible with some very basic logical skills; you can skip all the theorem proofs for his system KK):
  - Goosens, W. [1979]: ‘Alternative Axiomatizations of Elementary Probability Theory’. *Notre Dame J. of Formal Logic* XX(1).

Exercise

- Derive the following theorems from the axioms [P1] to [P3]. Theorems proved can figure in the derivation of subsequent ones. Solutions will be provided the session after next.

  (i) $0 \leq \Pr(P) \leq 1$

  (ii) $\Pr(P \cup Q) = \Pr(P) + \Pr(Q) - \Pr(P \cap Q)$

  (iii) $\Pr(P \cap Q) \leq \Pr(P)$

  (iv) $\Pr(P \cup Q) \geq \Pr(P)$

  (v) If $\Pr(P) = \Pr(Q) = 0$, then $\Pr(P \cup Q) = 0$

  (vi) If $\Pr(P) = 1$, then $\Pr(P \cap Q) = \Pr(Q)$

  (vii) If $P \subseteq Q$, then $\Pr(P \cap Q) = \Pr(P)$

  (viii) If $P \subseteq Q$, then $\Pr(P) \leq \Pr(Q)$