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[4] More Probability Theory

0. Outline

1. **Countable additivity (ctd.)**
2. **Conditional probability**
3. **The 'zero denominator' issue and alternative axiomatisations**

1. Countable additivity (ctd.)

- End of last lecture:
  - Discussion of the difference between the class of merely finitely additive probability functions and the narrower class of countably additive functions.
  - Mention of the controversy regarding whether belief functions are merely rationally required to be the former or are required to be the latter.
  - Part of De Finetti's 'guess the number' argument against rational belief functions = countably additive prob. functions:

    [1] It is rationally permissible for Paul to hold an equal degree of belief = d in each of the members of the countably infinite set $\{A_1, A_2, \ldots\}$ (where A_i = the set of p-worlds in which John is thinking of natural number i).

1. Countable additivity (ctd.)

[2] If [P1], [P2] and [P3*] are rational requirements on degrees of belief, then [1] is false.

- Here's the rest of the argument:

[3] If [P3*] isn't required and merely [P3] is required instead, then Paul *is* rationally entitled hold an equal degree of belief d in each of the aforementioned alternatives.

Proof: [P1]-[P3] allow him to have an equal degree of belief $d = 0$ in each alternative but a degree of belief of 1 in their union. ■

Note that these rules *don't* allow him to have an equal degree of belief $d > 0$ in each alternative but a degree of belief of 1 in their union!

1. Countable additivity (ctd.)

Proof: The 'Archimedean Principle' tells us that for any two positive real numbers x (e.g. d) and y (e.g. 1), there exists some natural number n such that $nx > y$ (e.g. $nd > 1$).

Therefore, were it to be the case that $d > 0$, there would exist a finite set of n alternatives $\{A_1,\ldots, A_n\}$, such that, by [P3], Paul's d.o.b in $\bigcup_{i=1}^{n} A_i$ would be rationally required to be > 1.

Problem: according to PROP, [T6] tells us that if Paul is rational and $P \subseteq Q$, he should be at least as confident in Q as he is in P. But $\bigcup_{i=1}^{n} A_i \subseteq \Omega$, hence Paul's d.o.b in Ω would be rationally required to be > 1, contradicting [P2]. ■

[4] Hence [P3*] needs to go in favour of [P3].

- There are a number of possible responses here.

1. Countable additivity (ctd.)

- One would be to challenge De Finetti on his first premise: why exactly *is* it rationally permissible for Paul to have an equal d.o.b in each alternative? (we might return to this issue later on)
- There is a *large* literature on the topic. Williamson [1999] discusses De Finetti's argument, gives relevant references and provides an argument in favour of requiring countable additivity.

2. Conditional probability

- So far: talk of S believing that P to degree d ($\text{Bel}_S(P) = d$; I omit reference to t).

  E.g.: I am 80% certain that I will win the race.
- Many theorists, however, sanction the existence of a further kind of cognitive state: believing that P to degree d *conditional on*, or *on the assumption that* Q ($\text{Bel}_S(P \mid Q)$).

  E.g.: I am 80% certain that I will win the race conditional on Fast Freddy's withdrawing from the competition.

  One (disputed but not-too-silly) way of understanding the notion: the value of $\text{Bel}_S(P \mid Q)$ = the value that $\text{Bel}_S(P)$ would have, were S to come to believe Q with absolute certainty.

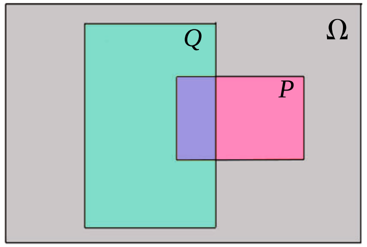
2. Conditional probability

- Bayesians argue that just like believing that P to degree d should obey the rules of the probability calculus, so too should its conditional counterpart: conditional degrees of belief should behave like *conditional probabilities*. So what are *they*?
- Note: the probability of P given Q is denoted $\Pr(P \mid Q)$.
- In the standard (Kolmogorov) axiomatisation, conditional probabilities are definitionally related to unconditional probabilities as follows:

  [P4] If $\Pr(Q) > 0$, then $\Pr(P \mid Q) =_{def} \dfrac{\Pr(P \cap Q)}{\Pr(Q)}$
- Note that it follows from this that if $\Pr(Q) = 0$, there is *no such thing* as $\Pr(P \mid Q)$, for any P (i.e. $\Pr(P \mid Q)$ is undefined, for any P). More on this feature of [P4] shortly.

2. Conditional probability

- The concept is easy to visualise on a Venn diagram: just as (i) Pr(P) is the ratio of the area of P to the area of Ω, (ii) Pr($P \mid Q$) is the ratio of the area of P to the area of Q, on condition that Q has an area ≠ 0.



- Note: if Pr(Q) > 0, all axioms and theorems for unconditional probabilities have a conditional counterpart in which each Pr(•) is replaced by Pr(• | Q) (i.e. conditional probabilities, when defined, behave like unconditional probabilities).

2. Conditional probability

- The analogues of axioms [P1]-[P3] :

  [T8] If Pr(Q) > 0, then Pr($P \mid Q$) ≥ 0.

  Proof: by [P4], Pr($P \mid Q$) = the ratio of Pr($P \cap Q$) to Pr(Q), both of which, by [P1], are positive. ■

  [T9] If Pr(Q) > 0, then Pr($\Omega \mid Q$) = 1.

  Proof: by [P4], Pr($\Omega \mid Q$) = the ratio of Pr($\Omega \cap Q$) to Pr(Q). But $\Omega \cap Q = Q$. ■

  [T10] If Pr(Q) > 0 and $P \cap R = \varnothing$, then Pr($P \cup R \mid Q$) = Pr($P \mid Q$) + Pr($R \mid Q$)

  Proof: by [P4], Pr($P \cup R \mid Q$) = the ratio of Pr(($P \cup R$)$\cap Q$) to Pr(Q). Since $P \cap R = \varnothing$, by [P3], Pr(($P \cup R$)$\cap Q$) = Pr($P \cap Q$) + Pr($R \cap Q$). By [P4], we recover the RHS of the equality. ■

2. Conditional probability

- Some further useful theorems:

  [T11] *Total Probability - Conditional Version :* if 0 < Pr(Q) < 1, then $\Pr(P) = \Pr(P|Q)\Pr(Q) + \Pr(P|\overline{Q})\Pr(\overline{Q})$

  Proof: this just falls straight out of [T7] and [P4]. ■

  [T12] *Bayes' Theorem I:* if Pr(H) > 0 and Pr(E) > 0, then

  $$\Pr(H|E) = \frac{\Pr(E|H)\Pr(H)}{\Pr(E)}$$

  Proof: successive application of [P4] to Pr($H \mid E$) then to Pr($H \cap E$). ■

  [T13] *Bayes' Theorem II:* if Pr(E) > 0 and 0 < Pr(H) < 1, then

  $$\Pr(H|E) = \frac{\Pr(E|H)\Pr(H)}{\Pr(E|H)\Pr(H) + \Pr(E|\overline{H})\Pr(\overline{H})}$$

2. Conditional probability

Proof: from [T12], by [T11]. ■

- Note: I have used the variables E and H here, rather than my usual P and Q. Why so?
- Because Bayes' Theorem is often used in the context of *scientific inference*, to compute the rational degrees of belief in a certain hypothesis (H) conditional on the evidence observed (E).
- Numerical example:
  - *Assume that I am rational.*
  - *My d.o.b. that any given 40yo woman who participates in routine screening has breast cancer = 0.01.*
  - *My d.o.b that someone ends up with a positive mammography given that she has breast cancer = 0.8.*

2. Conditional probability

- *My d.o.b that someone ends up with a positive mammography given that she doesn't have breast cancer = 0.096.*
- *What d.o.b do I have in x having breast cancer given that x is a 40yo woman who participates in routine screening and has had a positive mammography?*
- Let: *H* stand for breast cancer, *E* stand for a positive mammography.
- We have:
  - $\Pr(H) = 0.01$, and hence $\Pr(\overline{H}) = 0.99$
  - $\Pr(E|H) = 0.8$
  - $\Pr(E|\overline{H}) = 0.096$

2. Conditional probability

- We are after $\Pr(H|E)$.
- By *Bayes' Theorem (II)*:

$$\Pr(H|E) = \frac{\Pr(E|H)\Pr(H)}{\Pr(E|H)\Pr(H) + \Pr(E|\overline{H})\Pr(\overline{H})}$$

- Plugging in the relevant values:

$$\Pr(H|E) = \frac{0.8 \times 0.01}{0.8 \times 0.01 + 0.096 \times 0.99} \approx 0.078$$

- So according to PROB, the answer is roughly 0.078

3. The 'zero denominator' issue and alternative axiomatisations

- It is widely thought that Kolmogorov's axiomatisation has an awkward consequence when conjoined with PROB:

  It follows that if a rational agent has degree of belief = 0 in *Q* there is no conditional degree of belief in *P* given *Q* that he is rationally obligated to have ($\Pr(P | Q)$ comes out undefined).
- Many find this counterintuitive.
- Here's one sample argument…
- Think back to De Finetti's guess-the-number scenario, in which Paul has to guess which natural number John is thinking of.
- Say that, agreeing with De Finetti, we assume that rational degree of belief functions are merely finitely additive probability functions.

3. The 'zero denominator' issue and alternative axiomatisations

- So Paul *is* rationally entitled to assign equal degrees of belief *d* in each of the members of the countably infinite set $\{A_1, A_2, \ldots\}$ (where A_i = the set of p-worlds in which John is thinking of natural number *i*). Let's say he does just this.
- As we saw earlier, on pains of violating [P2], $d = 0$.
- Now consider the following propositions:

  A_2 = the set of p-worlds in which John picked the number 2.

  A_{EVEN} = the set of p-worlds in which John picked an even number.
- It seems to be the case that Paul is rationally obligated to have a conditional degree of belief d^* in A_{EVEN} given A_2 of 1.
- So we have a problem…

3. The 'zero denominator' issue and alternative axiomatisations

- Note: the zero-denominator issue isn't the only worry regarding [P4]; see Hajek [2003] for a whole battery of arguments.
- Various combinations of the following general properties of conditional probabilities (*irrespective* of whether the antecedent has an unconditional probability of 0) have been endorsed for at various times (Goosens [1979]):

  (i) $\Pr(P \mid P) = 1$

  (ii) If $Q \subseteq P$, then $\Pr(P \mid Q) = 1$

  (iii) $\Pr(P \mid Q) + \Pr(\overline{P} \mid Q) = 1$

  (iv) If $P \cap R = \emptyset$, then $\Pr(P \cup R \mid Q) = \Pr(P \mid Q) + \Pr(R \mid Q)$

  (v) If $Q \subseteq \overline{P}$, then $\Pr(P \mid Q) = 0$

  (Note: (ii) entails (i) and (iii) & (v) entails (ii)).

3. The 'zero denominator' issue and alternative axiomatisations

- Note: *on condition that* the antecedents of the conditional probabilities have an unconditional probability > 0, all these follow from [P1]-[P4]; we are worrying about what happens when this condition is violated.
- Now, as Goosens [1979:232] points out (i)-(v) are collectively inconsistent. Here, X marks the joint inconsistencies:.

	(i)	(ii)	(iii)	(iv)	(v)
(i)				X	X
(ii)			X	X	X
(iii)		X			X
(iv)	X	X			
(v)	X	X	X		

E.g.: (ii) (i.e. If $Q \subseteq P$, then $\Pr(P \mid Q) = 1$) $\models \neg$(iii) (i.e. $\Pr(P \mid Q) + \Pr(\overline{P} \mid Q) = 1$). *Proof*: since $\emptyset \subseteq P$ for any P, it follows from (ii) that $\Pr(P \mid \emptyset) = 1$ and $\Pr(\overline{P} \mid \emptyset) = 1$, and hence that (iii) is false. ■

Reference

- Goosens, W. [1979]: 'Alternative Axiomatizations of Elementary Probability Theory'. *Notre Dame J. of Formal Logic* XX(1).
- Hajek, A. [2003]: 'What Conditional Probability Could Not Be', *Synthese* 137(3): 273-323.
- Williamson, J. [1999]: 'Countable additivity and subjective probability', *British Journal for the Philosophy of Science* 50(3): 401-416.

Next lecture: 'Leftovers from Probability Theory + Synchronic Dutch books'

- No reading

Exercise

- Derive the following theorems from the axioms [P1] to [P4], *on the assumption* that the relevant conditional probabilities are well-defined. Solutions will be provided the session after next.

(i) $\Pr(\overline{P}|Q) = 1 - \Pr(P|Q)$

(ii) $\Pr(P \cap Q|R) = \Pr(P|R)\Pr(Q|P \cap R)$

(iii) $\Pr(P|P) = 1$

(iv) If $Q \subseteq P$, then $\Pr(P|Q) = 1$

(v) If $P \cap R = \varnothing$, then $\Pr(P \cup R|Q) = \Pr(P|Q) + \Pr(R|Q)$

(vi) If $Q \subseteq \overline{P}$, then $\Pr(P|Q) = 0$