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[5] Leftover Probability Theory

0. Outline

1. The ‘zero denominator’ issue and alternative axiomatisations (ctd.)
2. Independence

1. The ‘zero denominator’ issue and alternative axiomatisations

- Previous lecture:
  - We saw that some theorists are unhappy with the ‘ratio analysis’ of conditional probability (i.e. [P4]), notably because:
    - on that analysis, if $\Pr(Q) = 0$, $\Pr(P \mid Q)$ comes out as undefined, and…
    - if we are taking the line that rational degrees of belief should be probabilities, in the formal sense of the term, it then follows that if an agent has Bel(Q) for some Q, there is no such thing as his rational degree of belief in P given Q. But this seems wrong. We saw a scenario in which the rational degree of belief in question seemed to be 1.

1. The ‘zero denominator’ issue and alternative axiomatisations

- I then showed you a list of *general* statements that people have felt should be derivable from any adequate axiomatisation of conditional probability:
  - (i) $\Pr(P \mid P) = 1$
  - (ii) If $Q \subseteq P$, then $\Pr(P \mid Q) = 1$
  - (iii) $\Pr(P \mid Q) + \Pr(\overline{P} \mid Q) = 1$
  - (iv) If $P \cap R = \emptyset$, then $\Pr(P \cup R \mid Q) = \Pr(P \mid Q) + \Pr(R \mid Q)$
  - (v) If $Q \subseteq \overline{P}$, then $\Pr(P \mid Q) = 0$
- We saw that, surprisingly perhaps, these statements are jointly inconsistent (indeed quite a few are *pairwise* inconsistent!): you *can't* help yourself to all of them at once.
- How should we proceed then?

1. The 'zero denominator' issue and alternative axiomatisations

- Even if it isn't possible to consistently help oneself to *all* of (i)-(v), it *is* still possible to have consistent axiomatisations such that:
  - (i)-(v) are all true *on condition that* the antecedents of the conditional probabilities aren't contradictions (remember that when we saw that (ii)$\vDash\neg$(iii), the problem was created by Ø – the same is true of the other inconsistencies).
  - EITHER (in the exclusive sense) (i) & (ii) OR (v) are true *unconditionally* (i.e. *even* when the antecedents are contradictions)
- One example of an axiomatisation in which conditional versions of (i)-(v) come out as theorems is the 'slight modification' Carnap's axiomatisation (from Goosens [1979]).

1. The 'zero denominator' issue and alternative axiomatisations

- Let us say (slightly exegetically inaccurately) that Pr is a *Carnap probability function* iff it is a function from $\mathcal{F} \times \mathcal{F}$ to the reals, such that for all $P, Q, R \in \mathcal{F}$:

  [CP1] $\Pr(P \mid Q) \geq 0$

  [CP2] If $P \neq \varnothing$, then $\Pr(P \mid P) = 1$

  [CP3] If $Q \neq \varnothing$, then $\Pr(P \mid Q) + \Pr(\overline{P} \mid Q) = 1$

  [CP4] If $P \cap R \neq \varnothing$, then $\Pr(P \cap R \mid Q) = \Pr(R \mid P \cap Q)\Pr(P \mid Q)$

  [CP5] $\Pr(P) = \Pr(P \mid \Omega)$
- Notice here ([CP5]) that unconditional probabilities of the form $\Pr(P)$ come out as conditional probabilities of the form $\Pr(P \mid \Omega)$.

1. The 'zero denominator' issue and alternative axiomatisations

- It turns out that [CP1]-[CP5] entail [P1]-[P4]: i.e. if something is a Carnap probability function, it is also a Kolmogorov probability function. [P4] in particular (replacing '$=_{def}$' with '=' , of course) falls straight out of [CP4].
- Let's now move on to a central concept in probability theory, which we'll be needing later on when we discuss the issue of analysing the concept of evidential support.

2. Independence

- *Independence (binary)*: P and Q are probabilistically independent ('$P \perp\!\!\!\perp Q$') $=_{def}$ $\Pr(P \cap Q) = \Pr(P)\Pr(Q)$.
- Note that the following are equivalent to the RHS of the definition, providing the conditional probabilities are well-defined:
  - $\Pr(P \mid Q) = \Pr(P)$
  - $\Pr(Q \mid P) = \Pr(Q)$
  - $\Pr(Q \mid P) = \Pr(Q \mid \overline{P})$
  - $\Pr(P \mid Q) = \Pr(P \mid \overline{Q})$
- The proofs for the first and second pairs of equivalences are pretty much identical, so I'll stick to proofs for the 1st and 3rd equivalences.

2. Independence

- The proof for the first equivalence is trivial: $\Pr(P \cap Q) = \Pr(P)\Pr(Q)$ iff $\frac{\Pr(P \cap Q)}{\Pr(Q)} = \Pr(P)$, which by [P4] is equivalent to $\Pr(P \mid Q) = \Pr(P)$. ■
- The proof for the third equivalence is *slightly* longer:

  $\Pr(P \cap Q) = \Pr(P)\Pr(Q)$

  iff $\Pr(P \cap Q) = \Pr(P \cap \overline{Q})\Pr(Q) + \Pr(P \cap Q)\Pr(Q)$ (by [P3])

  iff $\Pr(P \cap Q) - \Pr(P \cap Q)\Pr(Q) = \Pr(P \cap \overline{Q})\Pr(Q)$ (algebra)

  iff $\frac{\Pr(P \cap Q)}{\Pr(Q)} = \frac{\Pr(P \cap \overline{Q})}{1 - \Pr(Q)}$ iff $\frac{\Pr(P \cap Q)}{\Pr(Q)} = \frac{\Pr(P \cap \overline{Q})}{\Pr(\overline{Q})}$ (by [T1]) iff $\Pr(P \mid Q) = \Pr(P \mid \overline{Q})$ (by [P4]). ■

2. Independence

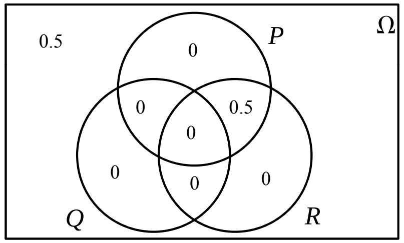
- $\perp\!\!\!\perp$, of course, is a binary relation. Let's take a look at some of its properties:
  - Symmetry. $\perp\!\!\!\perp$ is symmetric: for every model $\mathcal{M} = \langle \Omega, \mathcal{F}, \Pr \rangle$ and for every $P, Q \in \mathcal{F}$, if $P \perp\!\!\!\perp Q$ then $Q \perp\!\!\!\perp P$.

    Proof: $P \perp\!\!\!\perp Q =_{def} \Pr(P \cap Q) = \Pr(P)\Pr(Q)$. But $P \cap Q = Q \cap P$ and $\Pr(P)\Pr(Q) = \Pr(Q)\Pr(P)$. ■
  - Transitivity. $\perp\!\!\!\perp$ is nontransitive (i.e. neither transitive nor intransitive): for some model $\mathcal{M} = \langle \Omega, \mathcal{F}, \Pr \rangle$ and some P, Q, and $R \in \mathcal{F}$, $P \perp\!\!\!\perp Q$, $Q \perp\!\!\!\perp R$ but not $P \perp\!\!\!\perp R$.

    Proof : the next slide contains a probability model in which $P \perp\!\!\!\perp Q$, $Q \perp\!\!\!\perp R$ but not $P \perp\!\!\!\perp R$. ■

2. Independence

Here is the model in question:



Note: the probability function here is known as an '*irregular' probability function* (i.e. it assigns 0 or 1 to probabilities of propositions that $\neq \Omega$ and $\neq \emptyset$). Regular functions to the same effect *are* also available, but the values are rather more 'exotic'.

2. Independence

- If P and Q aren't probabilistically independent, they are probabilistically *correlated*.
- They are *positively* correlated iff $\Pr(P \cap Q) > \Pr(P)\Pr(Q)$, or equivalently (proofs omitted; have a go at home!):
  - $\Pr(P \mid Q) > \Pr(P)$
  - $\Pr(Q \mid P) > \Pr(Q)$
  - $\Pr(Q \mid P) > \Pr(Q \mid \overline{P})$
  - $\Pr(P \mid Q) > \Pr(P \mid \overline{Q})$
- If the inequality runs the other way, they are probabilistically *anti*-correlated.
- Unfortunately, I know of no 'official' symbol for correlation / anti-correlation. I will opt for $\not\perp\!\!\!\perp^{+}$ / $\not\perp\!\!\!\perp^{-}$.

2. Independence

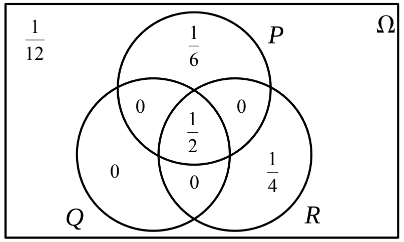
- Just like $\perp\!\!\!\perp$, $\not\perp\!\!\!\perp^{+}$ and $\not\perp\!\!\!\perp^{-}$ are binary relations. What are their properties?
  - Symmetry. Both relations are symmetric.

    Proof: Same proof as for symmetry of $\perp\!\!\!\perp$. ■
  - Transitivity. Both relations are non-transitive.

    Proof : the next slide contains a probability model in which $P \not\perp\!\!\!\perp^{+} Q$, $Q \not\perp\!\!\!\perp^{+} R$ but not $P \not\perp\!\!\!\perp^{+} R$. I omit the model pertaining to $\not\perp\!\!\!\perp^{-}$ (same principle). ■

2. Independence

Here is the model in question:



Note: regular functions to the same effect are again also available.

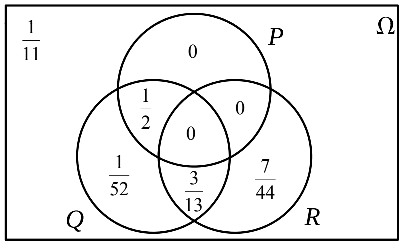
2. Independence

- *Conditional independence (binary)*: P and Q are independent conditional on R (‘$P \perp\!\!\!\perp Q \mid R$’) $=_{def}$ $\Pr(P \cap Q \mid R) = \Pr(P \mid R)\Pr(Q \mid R)$
- Here again, we have a number of equivalences for the RHS, which are simply the conditional counterparts of the equivalences for the definition of unconditional independence (e.g. $\Pr(P \mid Q \cap R) = \Pr(P \mid R)$, etc.).
- Similarly, conditional independence is both symmetric and non-transitive.
- *R screens P off from Q* $=_{def}$ R screens P off from Q $=_{def}$ $P \perp\!\!\!\perp Q \mid R$. This concept of screening off is widely used in philosophy.
- Note: we sometimes get: ‘R screens P off from Q $=_{def}$ $P \perp\!\!\!\perp Q \mid R$ *and* $P \perp\!\!\!\perp Q \mid \overline{R}$.’

2. Independence

- This is not an equivalent definition!

  Proof: Here is a model in which $P \perp\!\!\!\perp Q \mid R$ but not $P \perp\!\!\!\perp Q \mid \overline{R}$:



Note: regular functions to the same effect are again also available.

Reference

- Ramsey F. [1931]: ‘Truth and Probability’, in his *The Foundations of Mathematics*. London: Routledge.
- De Finetti, B. [1931]: ‘Sul significato soggestivo della probabilita’, *Fundamenta Mathematicae* XVII. Translated as “Foresight. Its Logical Laws, Its Subjective Sources”, in *Studies in Subjective Probability*, H. E. Kyburg, Jr. and H. E. Smokler (eds.), Robert E. Krieger Publishing Company, 1980.

Next lecture: ‘Synchronic Dutch Book Arguments’

- Reading:
  - Resnick, M. [1987]: *Choices: an introduction to decision theory.* Minneapolis:University of Minnesota Press. Section 3-3c ‘Subjective views’ and section 3-3d ‘Coherence & conditionalisation’ §1&2 *only*.
- Further reading (strongly recommended):
  - Hajek, A. [forthcoming]: ‘Dutch Book Arguments’, in P. Anand, P. Pattanaik, and C. Puppe (eds.) *The Oxford Handbook of Corporate Social Responsibility*.