Jake Chandler

Department of Philosophy, University of Glasgow,
67-69 Oakfield Avenue, Glasgow G12 8QQ
✉ J.Chandler@philosophy.arts.gla.ac.uk

UNIVERSITY of GLASGOW

[6] Synchronic Dutch Book Arguments

0. Outline

1. Taking stock

- Recap:
  - We set out to investigate various properties of rational belief.
  - We assumed that the objects of beliefs are propositions, and (somewhat controversially) that propositions are sets of possible worlds (L1).
  - In view of this, we went over some fundamentals of set theory, including operations on sets, cardinality, relations and functions (L1 & L2).
  - We argued that beliefs come in degrees, i.e. that there is a function $\text{Bel}_{S,t}$ that maps various propositions onto the corresponding degrees of belief of the subject *S* at time *t* (L1).

1. Taking stock

- We also countenanced the existence of conditional as well as unconditional degrees of belief (L4).
- We saw (L1) that a number of people argue in favour of:

  PROB: If *S* is rational at time *t*, $\text{Bel}_{S,t}$ is a probability function.
- In order to understand exactly what this proposal involves, we took a look at the mathematical concept of probability function.
  - We defined both finitely additive and countable additive probability functions (L3).
  - We noted the existence of a debate as to whether $\text{Bel}_{S,t}$ is rationally required to be an instance of the latter or merely the former (L3).

1. Taking stock

- We introduced the notion of conditional probability (L4), following Kolmogorov in adding [P4] to our existing stock of axioms [P1] - [P3]/[P3*].
- We noted that the conjunction of PROB and the Kolmogorov axiomatisation had some awkward consequences regarding rational conditional degrees of belief and took a look as some ways of getting round the issue (e.g. axiomatisations *à la* Carnap) (L4 & L5).
- We defined the technical notions of probabilistic independence and probabilistic correlation, noting a number of properties thereof (L5).
- Aside from the polemic regarding countable additivity, this was all pretty much non-philosophical groundwork...

1. Taking stock

- We finally turn to philosophy proper (!).
- First task: after explaining the *content* of PROB (what PROB *means*), we turn to its *justification*.
- There are a number of arguments adduced to support PROB.
- We'll start off with an extremely famous family of these:

  so-called 'Synchronic Dutch Book' arguments (aka SDBA's), versions of which were offered in the early 20th C by Ramsey [1931] and De Finetti [1931].

  (Why 'synchronic'? Because they concern constraints on an agent's d.o.b's *at a given time*, rather than over time.

  There are also 'diachronic' DBA's in favour of constraints on the updating of belief over time. More on this later.)

1. Taking stock

- Now, whatever the particular variety, Dutch book arguments proceed in two steps (here I try to be as general as possible):

  (i) Graded beliefs have certain properties, pertaining to facts about *betting behaviour* (in the broad sense of the term).

  (ii) Given that graded beliefs have these properties, agents are guilty of some particular kind of *betting-behaviour-related irrationality* if and only if their degrees of belief violate the rules of probability.
- So we need some betting lingo. Here is a brief overview of some terminology...

2. Some betting terminology

- Some definitions:
  - Let us say that a *bet* (gamble, wager,…) *B* regarding whether or not *P* is a situation in which a subject *S* gives up $q\mathcal{S}$ units of some good (e.g. money) in exchange for $\mathcal{S}$ units of an equivalent good if *P* is true and nothing if *P* is false.
  - In other words: if someone makes a bet on whether or not *P*, she will wind up with $\mathcal{S} - q\mathcal{S}$ if *P* and $- q\mathcal{S}$ if not.
  - We can represent such a bet as the ordered triple $\langle P, q, \mathcal{S}\rangle$ and denote the associated payoff with $\|\langle P, q, \mathcal{S}\rangle\|$.
  - Payoff table:

P	$\|\langle P, q, \mathcal{S}\rangle\|$
T	$\mathcal{S} - q\mathcal{S}$
F	$- q\mathcal{S}$

2. Some betting terminology

- We'll say that if $\mathcal{S} > 0$, the bet is *on P*; if $\mathcal{S} < 0$, the bet is *against P*. Rationale:
  - if $\mathcal{S}$ is strictly *positive*, *S* is strictly better off if *P* pans out *true* than if it pans out false (as $\mathcal{S} - q\mathcal{S} > - q\mathcal{S}$).
  - If $\mathcal{S}$ is strictly *negative*, *S* is strictly better off if *P* pans out *false* than if it pans out true (as $\mathcal{S} - q\mathcal{S} < - q\mathcal{S}$)
- Note that a bet, as defined here, needn't involve bookies or casinos!

  E.g: walking to the shop to buy some bread = betting on the shop being open, by paying the cost of walking ($- q\mathcal{S}$) in exchange for getting to buy a loaf of bread if the shop is open ($\mathcal{S}$), and getting to buy nothing if the shop is closed (0).

2. Some betting terminology

- $\mathcal{S}$ is known as the *stake* of the bet.
- *q* is known as the *betting quotient* (or betting rate/price) of the bet.
- People also sometimes characterise a bet in terms of its *odds*, which is simply the ratio of *q* to $(1 - q)$.
- Example: I pay the bookie a fiver in exchange for 20 pounds if Jammy Barnabas wins tomorrow's race at Walthamstow and nothing if he loses. I have made a bet on Jammy winning, with the following stakes/rates/odds:
  - $\mathcal{S} = 20$
  - $q\mathcal{S} = 20q = 5$, hence $q = 1/4$
  - odds = 1/3 (commonly written as '1:3')

2. Some betting terminology

- Note that the bookie has made a bet *against* Jammy winning, with the same betting rate, but a negative stake:
  - $\mathcal{S}^* = -20$ (he 'wins' -20 quid if Jammy wins)
  - $q\mathcal{S}^* = -20q = -5$ (he 'pays' -5 quid for the privilege), hence $q = 1/4$
  - odds = 1/3.
- A set of bets on or against various propositions (aka a *book*) which, if accepted, would guarantee a net loss, whatever the truth values of the propositions with respect to which the bets in the set were made, is known as a *Dutch* book.

3. Believing and betting: the classical line

- The classical version of the synchronic DBA (*à la* Ramsey / DeFinetti) takes possession of graded belief in a proposition to translate into dispositions to engage in certain kinds of betting behaviour
- Depending on the proponent of the argument, this belief-to-behaviour connection is viewed either as an empirical generalisation or as metaphysical/logical necessity.
- The connection in question is taken to be something like the following, where $|\mathcal{S}|$ denotes the absolute value of $\mathcal{S}$ ($|-5| = |5| = 5$):

  BET: For any agent *S*, and any proposition *P*, $Bel_S(P) = q$ iff for any $\mathcal{S}$, such that $|\mathcal{S}|$ is small with respect to *S*'s current welfare (more on this restriction shortly), *S* would accept betting arrangement $B = \langle P, q, \mathcal{S} \rangle$.

3. Believing and betting: the classical line

- Same thing in other words:

  BET: For any agent S, and any proposition P, $Bel_S(P) = q$ iff for any $\mathcal{S}$, such that $|\mathcal{S}|$ is small with respect to S's current welfare, S would accept to part with $q\mathcal{S}$ for a chance to win $\mathcal{S}$ if P and nothing if not.
- q is known as S's *fair betting quotient* for P.
- Example: I am – let's say – 50% confident in heads coming up at the next coin toss. According to BET:
  - I would happily part with 50 pence to obtain 1 pound if P and 0 pounds if not.
  - I would happily take 50 pence of someone on condition that I part with 1 pound if P and 0 pounds if not.

3. Believing and betting: the classical line

- Notice that it must be the case that q measures my degree of belief in P only if I would *both* (i) accept to bet *on P* and (ii) accept to bet *against P* at rate q.
- For instance:
  - whilst I *would* happily part with 0 pence ($- 0 \times \mathcal{S}$) to obtain 1 pound if a coin lands heads ($\mathcal{S}$) and 0 pounds if not,
  - I *wouldn't* happily take 0 pence of someone ($- 0 \times (- \mathcal{S})$) on condition that I part with 1 pound if the coin lands heads ($-\mathcal{S}$) and 0 pounds if not.
- So whilst I would accept a bet *on* heads at rate 0, according to BET, 0 doesn't measure my d.o.b in heads as I wouldn't be happy to accept a bet *against* heads at the same rate.

3. Believing and betting: the classical line

- Why require that $|\mathcal{S}|$ be small with respect to S's current welfare?
- Because, the thought goes, although I might be 50% confident in heads coming up at the next coin toss:
  - I *wouldn't* happily take out a loan to give up 30K pounds to obtain 60K pounds if P and 0 if not.
  - I wouldn't happily take 30K off you to risk landing myself a 60K debt if P.
- With this in hand, we can move on to step 2 of the argument:

  Assuming that graded belief translates into betting behaviour as outlined above in BET, S is guilty of some particular kind of *betting-behaviour-related irrationality* if and only if S's degrees of belief violate the rules of probability…

4. The Synchronic Dutch Book Theorem

- Step 2 hinges on two uncontroversial pieces of elementary maths. I'll take you through the proofs.
- Synchronic Dutch Book Theorem:

  DUTCH: assuming BET, if Bel_S doesn't satisfy [P1] – [P4], then S would accept a Dutch book.
- The proof of this is in four parts – one for each axiom (I'll assume $|\mathcal{S}|$ = relatively small throughout).
- **DUTCH$_1$**: assuming BET, if $Bel_S(P) = q < 0$, then Dutch book.

  Proof: assume $q < 0$. Then if $\mathcal{S} < 0$, then $q\mathcal{S} > 0$, $- q\mathcal{S} < 0$ and $\mathcal{S} - q\mathcal{S} < 0$. Therefore any single bet $\langle P, q, \mathcal{S}\rangle$, with $\mathcal{S} < 0$ (any bet against P) would guarantee S a loss, paying $\mathcal{S} - q\mathcal{S} < 0$ if P, and $- q\mathcal{S} < 0$ if $\overline{P}$. Dutch book. ■

4. The Dutch Book Theorem for PROB

E.g.: My degree of confidence in Jammy's wining the race is -1. According to BET, I therefore would accept to pay the bookie 5 pounds ($- q\mathcal{S} = 1 \times (-5) = -5$, with $\mathcal{S} = -5$) in exchange for paying him another 5 pounds if Jammy wins ($\mathcal{S} = -5$) and nothing if he loses. Dutch book.

- **DUTCH$_2$**: assuming BET, if $Bel_S(\Omega) = q \neq 1$, then Dutch book.

Proof: Here we break things into two: (i) if $q > 1$ then Dutch book and (ii) if $q < 1$ then Dutch book.

(i) Assume $q > 1$ and hence $(1 - q) < 0$. Then if $\mathcal{S} > 0$, then $\mathcal{S} - q\mathcal{S} = (1 - q)\mathcal{S} < 0$. Therefore any single bet $< \Omega, q, \mathcal{S}>$, with $\mathcal{S} > 0$ will guarantee S a loss, as it will pay out $\mathcal{S} - q\mathcal{S} < 0$ come what may (Ω is always true). Dutch book. ■

4. The Dutch Book Theorem for PROB

E.g.: My d.o.b in Jammy's either winning the race or not is 2. According to BET, I would accept to pay the bookie 10 pounds ($-q\mathcal{S} = -2 \times 5 = -10$, with $\mathcal{S} = 5$) in exchange for receiving 5 pounds if Jammy either wins or doesn't ($\mathcal{S} = 5$). Dutch book.

(ii) Now assume $q < 1$ and hence $(1 - q) > 0$. Then if $\mathcal{S} < 0$, then $\mathcal{S} - q\mathcal{S} = (1 - q)\mathcal{S} < 0$. Therefore any single bet $< \Omega, q, \mathcal{S}>$, with $\mathcal{S} < 0$ will guarantee S a loss, as it will pay out $\mathcal{S} - q\mathcal{S} < 0$ come what may (again, Ω is always true). Dutch book. ■

E.g.: My d.o.b in Jammy's either winning the race or not is only 0.5. According to BET, I would accept to receive 2.5 pounds from the bookie ($-q\mathcal{S} = -0.5 \times (-5) = 2.5$, with $\mathcal{S} = -5$) in exchange for paying him 5 pounds whether or not Jammy wins ($\mathcal{S} = -5$). Dutch book.

4. The Synchronic Dutch Book Theorem

- **DUTCH$_3$**: assuming BET, if $P \cap Q = \emptyset$ and $Bel_S(P \cup Q) = q_1 \neq Bel_S(P) + Bel_S(Q) = q_2 + q_3$, then Dutch book.

Proof: Again, we'll break this into two: (i) if $q_1 > q_2 + q_3$ then DB, (ii) if $q_1 < q_2 + q_3$ then DB.

This time, things are a little trickier. We'll have to consider a *combination* of bets:

a bet of whether P, plus

a bet on whether Q, plus

a bet on whether $P \cup Q$.

Let us set the stake of the first two bets to $\mathcal{S}$ and the stake of the third to $- \mathcal{S}$.

4. The Synchronic Dutch Book Theorem

Payoff table:

P	Q	$P \cup Q$	$\|<P, q_2, \mathcal{S}>\|$	$\|<Q, q_3, \mathcal{S}>\|$	$\|<P \cup Q, q_1, \mathcal{S}>\|$
T	T	T	$\mathcal{S} - q_2\mathcal{S}$	$\mathcal{S} - q_3\mathcal{S}$	$-(\mathcal{S} - q_1\mathcal{S})$
T	F	T	$\mathcal{S} - q_2\mathcal{S}$	$- q_3\mathcal{S}$	$-(\mathcal{S} - q_1\mathcal{S})$
F	T	T	$- q_2\mathcal{S}$	$\mathcal{S} - q_3\mathcal{S}$	$-(\mathcal{S} - q_1\mathcal{S})$
F	F	F	$- q_2\mathcal{S}$	$- q_3\mathcal{S}$	$q_1\mathcal{S}$

Note: I have shaded out the first line, as we are assuming that $P \cap Q = \emptyset$.

Assuming that the value of a set of bets is equal to the sum of the values of its components, we can calculate the payoff associated with accepting all the bets…

4. The Synchronic Dutch Book Theorem

If P is true and Q false, the payoff for the set of bets is:

$$\mathcal{S} - q_2\mathcal{S} - q_3\mathcal{S} - (\mathcal{S} - q_1\mathcal{S})$$
$$= \mathcal{S} - q_2\mathcal{S} - q_3\mathcal{S} - \mathcal{S} + q_1\mathcal{S}$$
$$= q_1\mathcal{S} - q_2\mathcal{S} - q_3\mathcal{S}$$
$$= \mathcal{S}(q_1 - (q_2 + q_3))$$

Payoff for other combinations of truth values are identical

(i) (i.e. if $q_1 < q_2 + q_3$ then DB) Assume $q_1 < q_2 + q_3$. Then if $\mathcal{S} > 0$, $\mathcal{S}(q_1 - (q_2 + q_3)) < 0$: the aforementioned combination of bets will guarantee S a loss, come what may, so long as $\mathcal{S} > 0$. Dutch book. ■

(ii) (i.e. if $q_1 > q_2 + q_3$ then DB) Assume $q_1 > q_2 + q_3$. Then if $\mathcal{S} < 0$, $\mathcal{S}(q_1 - (q_2 + q_3)) < 0$. Dutch book. ■

Next lecture: 'Synchronic Dutch Book Arguments (ctd.)'

- No set reading (but try working your way through the Hajek paper)

Exercise

- Prove the Dutch book theorems pertaining to the following theorems of probability theory (of course, I *don't* want to hear that these follow from the axioms and hence that the Dutch book theorems for these follow from the Dutch book theorems for the axioms):

  (i) $\Pr(\varnothing) = 0$

  (ii) $\Pr(P) = 1 - \Pr(\overline{P})$

  (iii) If $P \subseteq Q$, then $\Pr(P) \leq \Pr(Q)$