Elements of Deductive Logic

5. More on truth tables

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Truth tables for ‘complicated’ sentences

- During the last session, we took a look at the truth tables for the various connectives.
- With all these tables in hand, we can now construct truth tables for *any* kind of sentence whatsoever…

Truth tables for ‘complicated’ sentences: Example 1

- The construction of a table for $((p \equiv q)\&p) \supset q$:

p	q	$((p$	$\equiv$	$q)$	$\&$	$p)$	$\supset$	q

Step 1

Draw up a table: $r = 2^n$ rows, where n is the number of atomic sentences. (Here: $r = 4$.)

Column headers: atomic sentences on the left, complex sentence on the right.

Truth tables for ‘complicated’ sentences: Example 1 (ctd.)

- The construction of a table for $((p \equiv q)\&p) \supset q$:

p	q	$((p$	$\equiv$	$q)$	$\&$	$p)$	$\supset$	q
1	1							
1	0							
0	1							
0	0							

Step 2

Fill in the columns for the atomic sentences on the left:

- First column: alternate ‘1’ $\times r/2$ and ‘0’ $\times r/2$. (Here: $r/2 = 2$)
- Second column: alternate ‘1’ $\times r/4$ and ‘0’ $\times r/4$. (Here: $r/4 = 1$)
- …

Truth tables for ‘complicated’ sentences: Example 1 (ctd.)

- The construction of a table for $((p \equiv q) \& p) \supset q$:

p	q	$((p$	$\equiv$	$q)$	$\&$	$p)$	$\supset$	q
1	1	1	1	1	1	1	**1**	1
1	0	1	0	0	0	1	**1**	0
0	1	0	0	1	0	0	**1**	1
0	0	0	1	0	0	0	**1**	0

Step 3

Work your way up the parse tree, filling in the values for increasingly more complex sentences

- For atomic sentences: copy-paste the values under the letter
- For complex sentences: write the values under the main connective, using the relevant. tables

Truth tables for ‘complicated’ sentences: Example 2

- Let’s do another one: $((p \supset q) \& (q \supset r)) \& (p \& \sim r)$.

p	q	r	$((p$	$\supset$	$q)$	$\&$	$(q$	$\supset$	$r))$	$\&$	$(p$	$\&$	$\sim$	$r)$
1	1	1	1	1	1	1	1	1	1	**0**	1	0	0	1
1	1	0	1	1	1	0	1	0	0	**0**	1	1	1	0
1	0	1	1	0	0	0	0	1	1	**0**	1	0	0	1
1	0	0	1	0	0	0	0	1	0	**0**	1	1	1	0
0	1	1	0	1	1	1	1	1	1	**0**	0	0	0	1
0	1	0	0	1	1	0	1	0	0	**0**	0	0	1	0
0	0	1	0	1	0	1	0	1	1	**0**	0	0	0	1
0	0	0	0	1	0	1	0	1	0	**0**	0	0	1	0

Some important terms

- Each row of the table represents a valuation: an assignment of truth values to the argument’s sentences / subsentences that is consistent with the t. tables for the connectives.
- Formally, a valuation v is a function: it maps each sentence / subsentence onto a single truth value.
- If v assigns the value 1 to φ, we write $v(\varphi) = 1$. We then say that v satisfies, or is a model of, φ.
- If v assigns the value 0 to φ, we write $v(\varphi) = 0$, of course.
- Under the main connective of Example 1, we find a column of 1’s: the sentence is true on all valuations.
- The sentence is a tautology.

Some important terms (ctd.)

- Under the main connective of Example 2, we find a column of 0’s: the sentence is false on all valuations.
- The sentence is a contradiction.
- If a sentence is neither a tautology nor a contradiction, it is contingent.

Two more tables...

- Let's apply our procedure to two simple cases: $\sim(p\&\sim q)$ and $\sim p \vee q$.

p	q	$\sim$	$(p$	$\&$	$\sim$	$q)$
1	1	**1**	1	0	0	1
1	0	**0**	1	1	1	0
0	1	**1**	0	0	0	1
0	0	**1**	0	0	1	0

p	q	$\sim$	p	$\vee$	q
1	1	0	1	**1**	1
1	0	0	1	**0**	0
0	1	1	0	**1**	1
0	0	1	0	**1**	0

An interesting observation

- The two previous sentences have the same values under their main connectives.
- We say that they are logically equivalent to each other: $\sim(p\&\sim q) \Leftrightarrow \sim p \vee q$.
- Another way of putting this: $\sim(p\&\sim q) \equiv \sim p \vee q$ is a tautology.
- But they are also logically equivalent to *something else*, right?

p	q	$p \supset q$
1	1	1
1	0	0
0	1	1
0	0	1

The t. table for '⊃'

- These two equivalences highlight two lines of argument for the truth table for ⊃.
- Restall provides an argument involving $\sim(p\&\sim q)$:
  - If $\varphi \supset \psi$ is true, then $\varphi\&\sim\psi$ must be false and hence its negation, $\sim(\varphi\&\sim\psi)$, must be true.
  - Conversely, if $\sim(\varphi\&\sim\psi)$ is true, then if φ is true, then $\sim\psi$ must be false, and hence ψ must be true. So if $\sim(\varphi\&\sim\psi)$ is true, then $\varphi \supset \psi$ is also true.
  - Putting the two together, $\varphi \supset \psi$ and $\sim(p\&\sim q)$ are true in exactly the same circumstances: they must have the same truth tables.
- There is also a similar argument involving $\sim\varphi \vee \psi$.
- Later on in the course: arguments *against* our truth table for ⊃.

Truth tables for arguments

- We can apply the method of truth tables to prove the validity or invalidity of an argument form.
- We draw up, side by side, the sub-tables for the various $\mathcal{L}_S$ sentences of the argument form.
- Table for the disjunctive syllogism:

p	q	p	$\vee$	q	$\sim$	p	q
1	1	1	**1**	1	**0**	1	**1**
1	0	1	**1**	0	**0**	1	**0**
0	1	0	**1**	1	**1**	0	**1**
0	0	0	**0**	0	**1**	0	**0**

Truth tables for arguments (ctd.)

- We then consider the question:

  Is there a valuation v such that, for any premise φ, we have $v(\varphi) = 1$, but, for the conclusion ψ, we have $v(\psi) = 0$?

- If NO, then the form is valid.
- Where $\{\varphi_1, \ldots, \varphi_n\}$ is the set of premises and ψ the conclusion, we then write $\{\varphi_1, \ldots, \varphi_n\} \vDash \psi$.
- If YES, then the form is invalid.
- In this case, we write $\{\varphi_1, \ldots, \varphi_n\} \nvDash \psi$.

A valid form

- We can now see that the disjunctive syllogism is valid: $p \vee q, \sim p \vDash q$

p	q	p	$\vee$	q	$\sim$	p	q
1	1	1	**1**	1	**0**	1	**1**
1	0	1	**1**	0	**0**	1	**0**
0	1	0	**1**	1	**1**	0	**1**
0	0	0	**0**	0	**1**	0	**0**

- The only rows that give us a false conclusion are the second and fourth ones. But in both cases, one of the premises is false.

An invalid form

- An invalid form: denying the antecedent.
- We’ll show that $p \supset q, \sim p \nvDash \sim q$

p	q	p	$\supset$	q	$\sim$	p	$\sim$	q
1	1	1	**1**	1	**0**	1	**0**	1
1	0	1	**0**	0	**0**	1	**1**	0
0	1	0	**1**	1	**1**	0	**0**	1
0	0	0	**1**	0	**1**	0	**1**	0

- In Row 3, all the premises are true but the conclusion is false.
- The valuation that this row represents is known as a countermodel.

Next session

- Exercise class: please do exercise set #2