ELEMENTS OF DEDUCTIVE LOGIC

6. Tableau methods

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Two further uses of ‘$\models$’

- ‘$\models$’ is also used to talk about things that aren’t, strictly-speaking, *argument* forms.
- Two uses:
  (1) $\models \psi$: there is no valuation that does not satisfy ψ. In other words, ψ is a tautology.
  (2) $\{\varphi_1, \ldots, \varphi_n\} \models$: there is no valuation that satisfies every member of $\{\varphi_1, \ldots, \varphi_n\}$. We say that the set is inconsistent.
- Note that validity and inconsistency are interdefinable:
  $\{\varphi_1, \ldots, \varphi_n\} \models \psi$ iff $\{\varphi_1, \ldots, \varphi_n, \neg\psi\} \models$
- In English:
  An argument form is valid iff the set comprising the premises and the negation of the conclusion is inconsistent.

Two further uses of ‘$\models$’ (ctd.)

- The previous observation can also be recast as:
  An argument form is *in*valid iff there exists a valuation that satisfies each of the premises, as well as the negation of the conclusion.

Too many atoms spoil the broth

- The truth table method:
  For each possible assignment of values for the atoms, check whether the rules for the connectives force the premises to be true and the conclusion false.
- But things quickly get out of hand: 8 atoms $\Rightarrow 2^8 = 256$ rows!!
- Observation: we waste time by having to consider many assignments of values that turn out not to yield countermodels.
- The tableau, or tree, method avoids this issue:
  Assume the premises to be true and the conclusion to be false and check whether the rules for the connectives permit a compatible assignment of values for the atoms.
- This is a *top-down* rather than *bottom-up* approach.

Tableaux: the idea

- For tableaux, we in fact use the following equivalent procedure:

  Assume both the premises and the negation of the conclusion to be true and check whether the rules for the connectives permit a compatible assignment of values for the atoms.
- So how does this work in detail?
- Let us look at an example.

Tableaux by example: first case

Checking whether or not $p \supset q, r \vee \sim q \models (p \vee q) \supset r$

- A tableau consists of branches that each correspond to a set of would-be valuations.
- Writing φ on a branch represents the corresponding would-be valuations' assigning 1 to φ.

Tableaux by example: first case

Checking whether or not $p \supset q, r \vee \sim q \models (p \vee q) \supset r$

- All branches will contain the premises and the negation of the conclusion.
- We write these down at the root.

$$\begin{array}{c} p \supset q \\ r \vee \sim q \\ \sim ((p \vee q) \supset r) \end{array}$$

Tableaux by example: first case

Checking whether or not $p \supset q, r \vee \sim q \models (p \vee q) \supset r$

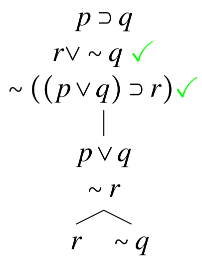
- Any branch that contains $\sim ((p \vee q) \supset r)$ will also contain both $p \vee q$ and $\sim r$.
- We tick off $\sim ((p \vee q) \supset r)$ and extend the tableau accordingly.

$$\begin{array}{c} p \supset q \\ r \vee \sim q \\ \sim ((p \vee q) \supset r) \checkmark \\ | \\ p \vee q \\ \sim r \end{array}$$

Tableaux by example: first case

Checking whether or not $p \supset q, r \vee \sim q \vDash (p \vee q) \supset r$

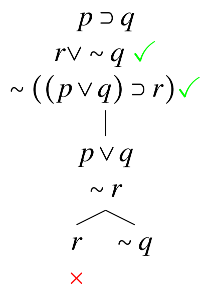
- Any branch that contains $r \vee \sim q$ will either contain r or contain $\sim q$.
- We tick off $r \vee \sim q$ and extend the tableau accordingly.



Tableaux by example: first case

Checking whether or not $p \supset q, r \vee \sim q \vDash (p \vee q) \supset r$

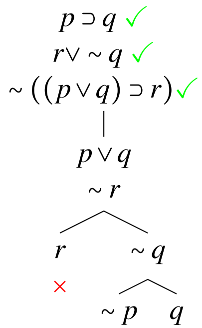
- Note that one branch contains both r and $\sim r$.
- But no valuation can assign 1 to both! This is a dead end.
- We say that the branch closes. Closed branches are marked by a ×.



Tableaux by example: first case

Checking whether or not $p \supset q, r \vee \sim q \vDash (p \vee q) \supset r$

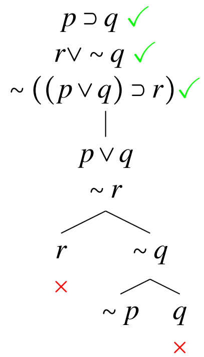
- Any branch that contains $p \supset q$ will either contain $\sim p$ or contain q.
- We tick off $p \supset q$ and extend the tableau accordingly.



Tableaux by example: first case

Checking whether or not $p \supset q, r \vee \sim q \vDash (p \vee q) \supset r$

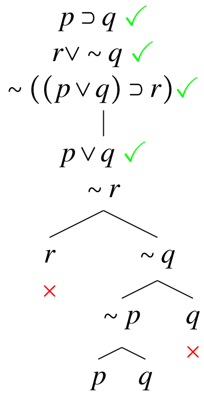
- Again, we have a dead end: one branch contains both $\sim q$ and q.
- We close the branch with a ×.



Tableaux by example: first case

Checking whether or not $p \supset q, r \vee \sim q \vDash (p \vee q) \supset r$

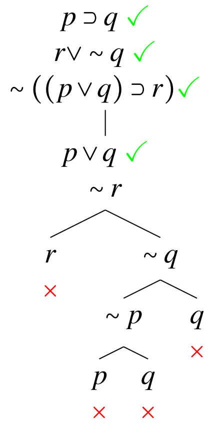
- Any branch that contains $p \vee q$ will either contain p or contain q.
- We tick off $p \vee q$ and extend the tableau accordingly.



Tableaux by example: first case

Checking whether or not $p \supset q, r \vee \sim q \vDash (p \vee q) \supset r$

- The remaining branches contain (i) $\sim p$ and p and (ii) $\sim q$ and q, respectively. They both close.
- All branches are now closed: the tableau itself is said to be closed.
- There is no valuation consistent with the truth of the premises and of the negation of the conclusion: the argument is *valid*.



Tableaux by example: second case

Checking whether or not $\vDash ((p \supset q)\&q) \supset p$

- All branches will contain $\sim (((p \supset q)\&q) \supset p)$, so we write it down.

$$\sim (((p \supset q)\&q) \supset p)$$

Tableaux by example: second case

Checking whether or not $\vDash ((p \supset q)\&q) \supset p$

- Any branch that contains $\sim (((p \supset q)\&q) \supset p)$ will contain both $((p \supset q)\&q)$ and $\sim p$.
- We tick off $\sim (((p \supset q)\&q) \supset p)$ and extend the tableau accordingly.

$$\sim (((p \supset q)\&q) \supset p) \checkmark$$
$$|$$
$$((p \supset q)\&q)$$
$$\sim p$$

Tableaux by example: second case

Checking whether or not $\vDash ((p \supset q)\&q) \supset p$

- Any branch that contains $((p \supset q)\&q)$ will contain both $p \supset q$ and q.
- We tick off $((p \supset q)\&q)$ and extend the tableau accordingly.

$\sim(((p \supset q)\&q) \supset p)$ ✓
|
$((p \supset q)\&q)$ ✓
$\sim p$
|
$p \supset q$
q

Tableaux by example: second case

Checking whether or not $\vDash ((p \supset q)\&q) \supset p$

- Any branch that contains $p \supset q$ will either contain $\sim p$ and contain q.
- We tick off $p \supset q$ and extend the tableau accordingly.

$\sim(((p \supset q)\&q) \supset p)$ ✓
|
$((p \supset q)\&q)$ ✓
$\sim p$
|
$p \supset q$ ✓
q
$\sim p$ $\quad$ q

Tableaux by example: second case

Checking whether or not $\vDash ((p \supset q)\&q) \supset p$

- We are left with two branches, both containing $\sim p$ and q.
- These branches are open. We mark them with a $\uparrow$.
- There exists a valuation that satisfies the negation of the sentence: the sentence is *not a tautology*.

$\sim(((p \supset q)\&q) \supset p)$ ✓
|
$((p \supset q)\&q)$ ✓
$\sim p$
|
$p \supset q$ ✓
q
$\sim p$ $\quad$ q
$\uparrow$ $\quad$ $\uparrow$

Next session

- Topic: more on tableaux.