Introduction to Logic

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Truth Tables

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Last week

- The language of sentential logic and its syntax
- How to translate an argument in English into sentential logic
- The issue of proving that a sentential form is valid as an instance of the SAT problem
- The notions of decidability and complexity

Introducing truth tables

Spotting a valid sentential form

- How do I *tell* that a sentence form is valid?
- Possibilities:
  (1) Notice that it is one of the well-know valid forms
  Problem: what if it isn't?
  (2) Brainstorm a counterexample
  Problem: very hard to do for non-trivial cases
- This week and next week: two easy and reliable methods
- Both involve searching for counterexamples *systematically*

Truth-functionality

- Our connectives are commonly taken to be:

  truth-functional: the truth value of their output is determined by (= is a function of) the truth value of their inputs.

- Swapping a subsentence for a subsentence with the same truth value shouldn't affect the truth value of the complex sentence

"…or …"

'*Melbourne is in Australia* or Melbourne is in France': true

'*Paris is in France* or Melbourne is in France': true

'*Humans have opposable thumbs* or Melbourne is in France': true

…

- *Not all* connectives are truth-functional!!

Truth-functionality (ctd)

"It's well known that…"

We have:

'It's well known that *Paris is the capital of France*': true

and:

'Paris is the capital of France': true

'The heart of a shrimp is located in its head': true

but:

'It's well known that *the heart of a shrimp is located in its head*': false!

Truth-functionality (ctd)

Other non truth-functional connectives

'It could have been the case that…'

'It's a pity that…'

'It's immoral that…'

'If it *had been* the case that …, then it *would have been* the case that…' ($\neq$ 'If it *is* the case that …, then it *is* the case that…'!)

- Can you think of examples showing that these aren't truth-functional?

Truth tables

- The way in which a certain connective determines the truth value of its output from the truth value of its inputs is summarised in a truth table, using 0 for false and 1 for true
- Note: here we assume there are only two truth values ($\Rightarrow$ sentences cannot be neither true nor false)
- Mathematically speaking, connectives are Boolean functions
- Straightforward: conjunctions, negations
- Require comment: disjunctions, conditionals, maybe biconditionals
- With all these tables in hand, we'll then be able to construct tables for *any* kind of sentence or indeed argument whatsoever.

Conjunctions and negations

- A conjunction is true if and only if both its conjuncts are true. Otherwise it is false.

P	Q	$P\&Q$
1	1	1
1	0	0
0	1	0
0	0	0

- A negation is true if and only if its negand is false. Otherwise it is false.

P	$\sim P$
1	0
0	1

Biconditionals

- A biconditional is true if and only if both its terms have the same truth value. Otherwise it is false.

P	Q	$P \equiv Q$
1	1	1
1	0	0
0	1	0
0	0	1

Disjunctions

- A disjunction is true if and only if at least one of its disjuncts is true. Otherwise it is false.

P	Q	$P \vee Q$
1	1	1
1	0	1
0	1	1
0	0	0

- Note: a disjunction is considered true when both disjuncts are true (more on this shortly)

Conditionals

- A conditional is true if and only if either its antecedent is false or its consequent is true. It is false if and only if its antecedent is true but its consequent is false.

P	Q	$P \supset Q$
1	1	1
1	0	0
0	1	1
0	0	1

- Note: a conditional whose antecedent is false is considered true!
- This is a bit controversial. More on this shortly.

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Tables for longer sentences: Example 1

- The construction of a table for $((P \equiv Q) \& P) \supset Q$:

P	Q	$((P$	$\equiv$	$Q)$	$\&$	$P)$	$\supset$	Q

Step 1

Draw up a table: $r = 2^n$ rows, where n is the number of atomic sentences. (Here: $r = 4$.)

Column headers: atomic sentences on the left, complex sentence on the right.

Tables for longer sentences: Example 1 (ctd)

P	Q	$((P$	$\equiv$	$Q)$	$\&$	$P)$	$\supset$	Q
1	1							
1	0							
0	1							
0	0							

Step 2

Fill in the columns for the atomic sentences on the left, making sure to write down every possible combination of truth values.

Tables for longer sentences: Example 1 (ctd)

P	Q	$((P$	$\equiv$	$Q)$	$\&$	$P)$	$\supset$	Q
1	1		1					
1	0		0					
0	1		0					
0	0		1					

Step 3

Work your way up the parse tree, filling in the values for increasingly more complex sentences, writing the values under the main connective

Tables for longer sentences: Example 1 (ctd)

P	Q	$((P$	$\equiv$	$Q)$	$\&$	$P)$	$\supset$	Q
1	1		1		1			
1	0		0		0			
0	1		0		0			
0	0		1		0			

Step 3

Work your way up the parse tree, filling in the values for increasingly more complex sentences, writing the values under the main connective

Tables for longer sentences: Example 1 (ctd)

P	Q	$((P$	$\equiv$	$Q)$	$\&$	$P)$	$\supset$	Q
1	1		1		1		**1**	
1	0		0		0		**1**	
0	1		0		0		**1**	
0	0		1		0		**1**	

Step 3

Work your way up the parse tree, filling in the values for increasingly more complex sentences, writing the values under the main connective

Tables for longer sentences: Example 2

- Let's do another one: $((P \supset Q)\&P)\& \sim Q$.

P	Q	$((P$	$\supset$	$Q)$	$\&$	$P)$	$\&$	$\sim$	Q

Tables for longer sentences: Example 2 (ctd)

P	Q	$((P$	$\supset$	$Q)$	$\&$	$P)$	$\&$	$\sim$	Q
1	1								
1	0								
0	1								
0	0								

Tables for longer sentences: Example 2 (ctd)

P	Q	$((P$	$\supset$	$Q)$	$\&$	$P)$	$\&$	$\sim$	Q
1	1		1						
1	0		0						
0	1		1						
0	0		1						

Tables for longer sentences: Example 2 (ctd)

P	Q	$((P$	$\supset$	$Q)$	$\&$	$P)$	$\&$	$\sim$	Q
1	1		1		1				
1	0		0		0				
0	1		1		0				
0	0		1		0				

Tables for longer sentences: Example 2 (ctd)

P	Q	$((P$	$\supset$	$Q)$	$\&$	$P)$	$\&$	$\sim$	Q
1	1		1		1			0	
1	0		0		0			1	
0	1		1		0			0	
0	0		1		0			1	

Tables for longer sentences: Example 2 (ctd)

P	Q	$((P$	$\supset$	$Q)$	$\&$	$P)$	$\&$	$\sim$	Q
1	1		1		1		**0**	0	
1	0		0		0		**0**	1	
0	1		1		0		**0**	0	
0	0		1		0		**0**	1	

Some important terms

- Each row of the table represents a valuation (aka evaluation, interpretation, model): an assignment of truth values to the argument's sentences / subsentences that is consistent with the t. tables for the connectives.
- Think of these as describing possible situations (aka possible worlds)
- Formally, a valuation v is a function: it maps each sentence onto a single truth value.
- If v assigns the value 1 to P, we write $v(P) = 1$. We then say that v satisfies, or is a model of, P.
- If v assigns the value 0 to P, we write $v(P) = 0$, of course.

Tautologies and contradictions

- Under the main connective of Example 1, we find a column of 1's: the sentence is true on all valuations
- The sentence is a tautology
- Under the main connective of Example 2, we find a column of 0's: the sentence is false on all valuations
- The sentence is a contradiction
- If we have a mix of 0's and 1's, we have a contingent sentence

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Expressive completeness

Tables for "exotic" connectives

- We have seen tables for 5 connectives: 1 "unary" and 4 "binary"
- But there are as many possible t. tables as there are possible combinations of truth values in the right hand cells
- For binary connectives: 4 cells, so $2^4 = 16$, and hence 12 more!
- For unary connectives: 2 cells, so $2^2 = 4$, so 3 more!
- These typically don't correspond to "real" (lexicalised) connectives

Tables for "exotic" connectives (ctd)

Some alternative truth tables

P	Q	$P \uparrow Q$
1	1	0
1	0	1
0	1	1
0	0	1

Alternative denial
(NAND; "not both")

P	Q	$P \downarrow Q$
1	1	0
1	0	0
0	1	0
0	0	1

Joint denial
(NOR; "neither nor")

P	Q	$P \underline{\vee} Q$
1	1	0
1	0	1
0	1	1
0	0	0

Exclusive or
(XOR; "either or but not both")

Expressing connectives in terms of others

- These connectives can be expressed using combinations of the standard ones

Expressing $\underline{\vee}$

P	Q	$P \underline{\vee} Q$
1	1	0
1	0	1
0	1	1
0	0	0

P	Q	$(P$	$\vee$	$Q)$	$\&$	$\sim$	$(P$	$\&$	$Q)$
1	1		1		**0**	0		1	
1	0		1		**1**	1		0	
0	1		1		**1**	1		0	
0	0		0		**0**	1		0	

Logical equivalence

- We say that $P \underline{\vee} Q$ and $(P \vee Q) \& \sim (P \& Q))$ are logically equivalent to each other:

  $P \underline{\vee} Q \Leftrightarrow (P \vee Q) \& \sim (P \& Q))$
- Two other ways of putting this:

  $P \underline{\vee} Q \equiv (P \vee Q) \& \sim (P \& Q))$ is a tautology

  $P \underline{\vee} Q$ and $(P \vee Q) \& \sim (P \& Q))$ entail each other

Expressive completeness

- A set of connectives is expressively complete if and only if you can use it to express any possible t. table
- It is minimally complete if and only if it doesn't have a subset that is complete.
- Our set $\{\vee, \&, \supset, \equiv, \sim\}$ is complete but not minimally so!
- Minimally complete sets using connectives that we've seen: $\{\downarrow\}$, $\{\uparrow\}$, $\{\vee, \sim\}$, $\{\&, \sim\}$, $\{\supset, \sim\}$
- Note that none of these involve $\equiv$ or $\veebar$

Two more tables...

- Let's apply our procedure to $\sim(P \& \sim Q)$ and $\sim P \vee Q$.

P	Q	$\sim$	$(P$	$\&$	$\sim$	$Q)$
1	1	**1**		0	0	
1	0	**0**		1	1	
0	1	**1**		0	0	
0	0	**1**		0	1	

P	Q	$\sim$	P	$\vee$	Q
1	1	0		**1**	
1	0	0		**0**	
0	1	1		**1**	
0	0	1		**1**	

- So $\sim(P \& \sim Q)$ and $\sim P \vee Q$ are equivalent

An interesting observation (ctd.)

- But they are also equivalent to *something else*, right?

P	Q	$P \supset Q$
1	1	1
1	0	0
0	1	1
0	0	1

- OK...and so what?

The "paradoxes of material implication"

- The t. table for $\supset$ is controversial
  - Agreement that $P \supset Q$ is false when P is true and Q is false
  - Some disagreement on the rest
- Indeed, it's easy to show that it has some arguably (!) awkward consequences
  (i) P entails $Q \supset P$

  Cats

  "Cats have tails. Therefore if cats have paws, then cats have tails."

  (ii) $\sim Q$ entails $Q \supset P$

  Wings

  "Bicycles don't have wings. Therefore if bicycles have wings, then fridges have wings too."

The “paradoxes of material implication” (ctd)

Also:

(iii) $(Q\&P) \supset R$ entails $(Q \supset R) \vee (P \supset R)$

> *Explosion*
>
> “If you leave the gas on and light a match, then the flat will blow up Therefore either the flat will blow up if you leave the gas on, or it will blow up if you light a match.”

(iv) $(Q \supset P) \vee (P \supset R)$ is a tautology

> *Beach*
>
> “Either we will go to the beach if it snows or pigs will fly if we go to the beach.”

- Our earlier equivalences can be used as a basis for an argument for the t. table for $\supset$

The t. table for ‘$\supset$’

- Restall provides an argument involving $\sim(P\& \sim Q)$:
  - If $P \supset Q$ is true, then $P\& \sim Q$ must be false and hence its negation, $\sim(P\& \sim Q)$, must be true.
  - Conversely, if $\sim(P\& \sim Q)$ is true, then if P is true, then $\sim Q$ must be false, and hence Q must be true. So if $\sim(P\& \sim Q)$ is true, then $P \supset Q$ is also true.
  - Putting the two together, $P \supset Q$ and $\sim(P\& \sim Q)$ are true in exactly the same circumstances: they must have the same truth tables.
- There is also a similar argument involving $\sim P \vee Q$.

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Truth tables for arguments

Truth tables for arguments

- We can apply the method of truth tables to prove the validity or invalidity of a sentential argument form.
- We draw up, side by side, the sub-tables for the various sentences of the argument form.
- Table for the disjunctive syllogism (see *Cluedo* example):

P	Q	$P \vee Q$	$\sim P$	Q
1	1	**1**	**0**	**1**
1	0	**1**	**0**	**0**
0	1	**1**	**1**	**1**
0	0	**0**	**1**	**0**

Truth tables for arguments (ctd.)

- We then consider the question:
  Is there a valuation that makes all the premises true but the conclusion false?
- If NO, then the form is valid.
- If YES, then the form is invalid.
- Metalinguistic symbol for validity: $\models$ (invalidity: $\not\models$)

Using the symbol for (in)validity

$p \models p \vee q$

$p \vee q \not\models p$

Decidability and complexity

- Good news: since the t. table method finds the answer in a finite number of steps, SAT is decidable
- Bad news: in general, our t. table procedure is complex!
- For n atomic sentences, we have 2^n valuations to check: exponential time!

Two further uses of '$\models$'

- '$\models$' is also used to talk about things that aren't, strictly-speaking, *argument* forms.
- Two uses:
  (1) $\models C$: there is no valuation that does not satisfy C. In other words, C is a tautology.
  (2) $P_1, \ldots, P_n \models$: there is no valuation that satisfies every member of $\{P_1, \ldots, P_n\}$. We say that the set is inconsistent (the conjunction of its members is a contradiction).

A valid form

- We can now see that the disjunctive syllogism is valid

P	Q	P	$\vee$	Q	$\sim$	P	Q
1	1		**1**		**0**		**1**
1	0		**1**		**0**		**0**
0	1		**1**		**1**		**1**
0	0		**0**		**1**		**0**

- The only rows that give us a false conclusion are the second and fourth ones. But in both cases, one of the premises is false
- Exercise: use tables to check the validity of our other valid sentential forms

Some invalid forms: *Affirming a Disjunct*

- Recall our examples of common arguments with deductively invalid sentential forms (formal fallacies)

Treats

(1) Isaac either got a cookie or got an ice-cream.

(2) Isaac got a cookie.

---

(3) Isaac didn't get an ice-cream.

Form: $P \vee Q$, P, therefore $\sim Q$ (aka Affirming a Disjunct)

Some invalid forms: *Affirming a Disjunct* (ctd)

P	Q	P	$\vee$	Q	P	$\sim$	Q
1	1		**1**		**1**	**0**	
1	0		**1**		**1**	**1**	
0	1		**1**		**0**	**0**	
0	0		**0**		**0**	**1**	

- Row 1 (P and Q both true): all the premises are true but the conclusion is false
- The valuation that this row represents is known as a countermodel (or counterexample)
- In *Treats*: Isaac's having both a cookie *and* an ice-cream

Some invalid forms: *Affirming the Consequent*

Steroids

(1) If steroids were taken, the test will come out positive.

(2) Sam's test came out positive.

---

(3) Sam took steroids.

Form: $P \supset Q$, Q, therefore P (aka Affirming the Consequent)

Some invalid forms: *Affirming the Consequent* (ctd)

P	Q	$P \supset Q$	Q	P
1	1	**1**	**1**	**1**
1	0	**0**	**0**	**1**
0	1	**1**	**1**	**0**
0	0	**1**	**0**	**0**

- Row 3 (P false, Q true): all the premises are true but the conclusion is false
- In *Steroids*: Sam's test came out positive, but he didn't take steroids (he's perhaps on some other medication)

Some invalid forms: *Denying the Antecedent*

Penknife

(1) If Billy's dad found his penknife, he's in deep trouble.

(2) Billy's dad didn't find his penknife.

---

(3) Billy isn't in deep trouble.

Form: $P \supset Q$, $\sim P$, therefore $\sim Q$ (aka denying the antecedent)

Some invalid forms: *Denying the Antecedent* (ctd)

P	Q	$P \supset Q$	$\sim P$	$\sim Q$
1	1	**1**	**0**	**0**
1	0	**0**	**0**	**1**
0	1	**1**	**1**	**0**
0	0	**1**	**1**	**1**

- Row 3 again (P false, Q true): all the premises are true but the conclusion is false
- In *Penknife*: Billy's dad didn't find the penknife, but Billy is still in trouble (for some other reason)

What is said vs what is implied

- But *why* do we make these mistakes?!
- Same arguable root cause:

  We are confusing what the premises *imply* with what the premises *say* and hence considering different arguments (which happen to be valid)

Interview

You tell me that you think the candidate was funny and smart. I tell you "I definitely think he was funny". I have told you one thing, but implied something rather stronger: that I think the candidate was funny *and that I don't think that the candidate was smart*.

What is said vs what is implied (ctd)

Greedy Dad

You ask me whether I've had some cookies yet. I tell you "Yes, actually, I have had some cookies.". I have said one thing but implied something stronger: that I have had some of the cookies *and that I have not had all of them*.

Back to our fallacies: Asserting a Disjunct

- When we read $P \vee Q$, we might assume that it isn't the case that both P and Q are true

  Interpretation: $P \underline{\vee} Q$
- On the alternative truth table, the argument is valid:

P	Q	P	$\underline{\vee}$	Q	P	$\sim$	Q
1	1		**0**		**1**	**0**	
1	0		**1**		**1**	**1**	
0	1		**1**		**0**	**0**	
0	0		**0**		**0**	**1**	

Back to our fallacies: The remaining two fallacies

- When we read $P \supset Q$, we might assume that there is no other sufficient condition for Q other than P

  Interpretation: $P \equiv Q$
- On the alternative truth table (here for Affirming the Consequent), the argument is valid:

P	Q	P	$\equiv$	Q	Q	P
1	1		**1**		**1**	**1**
1	0		**0**		**0**	**1**
0	1		**0**		**1**	**0**
0	0		**1**		**0**	**0**