Introduction to Logic

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Beyond Truth and Falsity, Pt. 2

LA TROBE UNIVERSITY

Last week

- The idea of trivalent logics, which allow for sentences to have a third value: neither true nor false (#)
- Different reasons to claim that these "truth value gaps" exist
- A first very simple proposal regarding how to define validity and adapt the classical truth tables to the trivalent case: the logic B3

Some Gappy Logics, Pt 2

The logic K3

- As with B3, $D = \{1\}$
- Again, the tables extend the bivalent ones:

$\sim$	
1	0
0	1
#	#

$\vee$	1	0	#
1	1	1	1
0	1	0	#
#	1	#	#

&	1	0	#
1	1	0	#
0	0	0	0
#	#	0	#

$\supset$	1	0	#
1	1	0	#
0	1	1	1
#	1	#	#

$\equiv$	1	0	#
1	1	0	#
0	0	1	#
#	#	#	#

$\Rightarrow$ The output is #, unless there are classical inputs that would classically suffice to determine a different output.

Comments on K3

- Is K3 any better than B3 for borderline cases? Future contingents?
- Recall:

Pirate/Banker

(10) 'I am a Somali pirate and there will be a sea battle tomorrow.'

(11) ' Either James is bald or he is a banker' (assuming James is a borderline bald banker)

- According to B3: both have value #
- According to K3: (10) has value 0 and (11) has value 1
- But some feel that problems remain…

Comments on K3 (ctd.)

Storm/Dot

(12) 'Either there will be a storm tomorrow or there won't'

(13) 'If the dot is green, then it is green'

- Some feel these are tautologies (how about you?)
- However (Priest 2008, p. 124):
  - $\not\models_{K3} \varphi \vee \sim \varphi$ and (equivalently)
  - $\not\models_{K3} \varphi \supset \varphi$.

  Countermodel: $v(\varphi) = \#$

Comments on K3 (ctd.)

- Special case of a weird general feature of K3:

  For all sentences φ, $\not\models_{K3} \varphi$

  In other words: no tautologies in K3!
- Indeed:
  - For any sentence φ, if all its atomic subsentences have value #, then so too does φ
  - So for any sentence φ, there is at least one valuation v such that $v(\varphi)$ is the non-designated value #
- This is also true of B3

The logic SV

- As before, $D = \{1\}$.
- How about truth tables?

  The connectives are not truth-functional in general!
- The truth value of a complex sentence is determined by the truth value of its subsentences only when these have classical values.
- So how does it work in general?
- To see how a valuation v assigns values to complex sentences, we first see what it does to the atomic sentences
- Let v_a be the restriction of v to the atomic sentences

Resolutions

- A resolution v_a^+ of v_a is a bivalent valuation that

  (i) agrees with v_a on its assignment of classical values but

  (ii) substitutes a classical value for every non-classical value assigned by v_a
- It fills in the gaps left in v_a

Resolutions

Let v_a be given by:	Let v_a^+ be given by:
$v_a(p) = 1$	$v_a^+(p) = 1$
$v_a(q) = \#$	$v_a^+(q) = 1$
$v_a(r) = \#$	$v_a^+(r) = 0$

Then v_a^+ is a resolution of v_a

Truth values in SV

- We now determine how v handles complex sentences, as follows:

  $v(\varphi) = 1$ iff $v_a^+(\varphi) = 1$ for all resolutions v_a^+ of v_a

  $v(\varphi) = 0$ iff $v_a^+(\varphi) = 0$ for all resolutions v_a^+ of v_a

  $v(\varphi) = \#$ in the remaining case

Truth values in SV

Let $v(\varphi) = \#$. What is (i) $v(\varphi \vee \sim \varphi)$ and (ii) $v(\varphi \vee \varphi)$?

For all resolutions v_a^+ of v_a, either

- $v^+(\varphi) = 1$ and $v^+(\sim \varphi) = 0$, or
- $v^+(\varphi) = 0$ and $v^+(\sim \varphi) = 1$

In both cases, $v^+(\varphi \vee \sim \varphi) = 1$, so $v(\varphi \vee \sim \varphi) = 1$

In the first case $v^+(\varphi \vee \varphi) = 1$ but in the second $v^+(\varphi \vee \varphi) = 0$, so $v(\varphi \vee \varphi) = \#$

Failure of truth-functionality in SV

- The previous example demonstrates the failure of truth-functionality for the connectives of SV:
  - In both cases the disjuncts have value #
  - In one case ($\varphi \vee \sim \varphi$), the disjunction has value 1
  - In the other case ($\varphi \vee \varphi$), it has value #

$\Rightarrow$ The value of the inputs isn't enough to determine the value of the outputs!

Validity in SV

- In section on K3: some controversial invalidities (e.g. $\not\models_{K3} \varphi \vee \sim \varphi$).
- What about here?
- An argument is SV-valid iff it is classically valid!
- For simple proof, see Priest (2008, p. 134).
- So in particular, $\models_{SV} \varphi \vee \sim \varphi$

Complete De Finettian tables for ⊃

- The tables for B3 and K3 extend the classical bivalent one
- But we've seen a partial table for ⊃ that revises it
- This partial table has been completed in a few ways, e.g.:

⊃	1	0	#
1	1	0	#
0	#	#	#
#	#	#	#

De Finetti (DF)

⊃	1	0	#
1	1	0	#
0	#	#	#
#	1	0	#

Cooper-Cantwell (CC)

- DF: if $v(\varphi) = \#$, then $v(\varphi \supset \psi) = \#$
- CC: if $v(\varphi) = \#$, then $v(\varphi \supset \psi) = v(\psi)$

Complete De Finettian tables for ⊃ (ctd)

- Baratgin *et al*: DF table best fits subjects intuitions
- Egré *et al* look at the validities obtained by pairing different tables with different trivalent definitions of validity (and favour CC paired with $D = \{1, \#\}$)

The paradoxes of material implication

- How do the paradoxes of material implication pan out with DF/CC (with $D = \{1\}$ and using K3 for & and ∨)?

(i) $\psi \models \varphi \supset \psi$
Invalid: let $v(\varphi) = 0$ and $v(\psi) = 1$

(ii) $\neg\varphi \models \varphi \supset \psi$
Invalid: let $v(\varphi) = 0$ and $v(\psi) = 1$

(iii) $(\varphi \& \psi) \supset \chi \models (\varphi \supset \chi) \vee (\psi \supset \chi)$
Valid

(iv) $\models (\varphi \supset \psi) \vee (\psi \supset \chi)$
Invalid: let $v(\varphi) = 1$, $v(\psi) = 0$ and $v(\chi) = 0$

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Adapting the Tableau Rules

Introduction

- How do we adapt the tableau method to trivalent frameworks?
- SV validates same inferences as classical logic: no need for new methods there
- To illustrate the situation for other logics, we'll consider K3
- These systems are demonstrably sound and complete (proof omitted; see Priest)
- We'll only consider the sentential case, although the case of predicate logic can also be easily handled

Roots

- Bivalent context:

  Find valuations such that the premises are true and the conclusion is false (equivalently: the negation of the conclusion is true)

- So to evaluate $\varphi_1, \ldots, \varphi_n \vdash \psi$, the root of the tree is:

$$\begin{array}{c} \varphi_1 \\ \ldots \\ \varphi_n \\ \sim \psi \end{array}$$

Roots (ctd)

- Multivalent context:

  Find valuations such that the premises have a designated value and the conclusion has a non-designated value.

- To evaluate $\varphi_1, \ldots, \varphi_n \vdash \psi$, the root of the tree is:

$$\begin{array}{c} \varphi_1, + \\ \ldots \\ \varphi_n, + \\ \psi, - \end{array}$$

- For K3 (also B3):
  - + = designated' (1)
  - - = non-designated (0 or #)

Closing rules

- The closing rules also change
- Bivalent context: the tableau closes iff we have, for any sentence φ, both φ and $\sim \varphi$ on the same branch.
- Here: the branch closes iff we have, for any sentence φ, either
  - (i) both $\varphi, +$ and $\varphi, -$ on the same branch *or* (since no sentence can be assigned both 1 and either 0 or #, because valuations only assign one value)
  - (ii) both $\varphi, +$ and $\sim \varphi, +$ on the same branch (since no sentence can be such that both it and its negation are assigned 1, because of the truth tables for negation)

Countermodels

- Countermodels: if a branch b fails to close, for every atomic sentence φ
  - if $\varphi, +$ is on b, then $v(\varphi) = 1$
  - if $\sim \varphi, +$ is on b, then $v(\varphi) = 0$
  - Otherwise $v(\varphi) = \#$

  Note: we never have both $\varphi, +$ and $\sim \varphi, +$ on an open branch.
- Tip: if you have a *prima facie* countermodel, double-check that it *is* indeed one, using the truth tables.

Resolving rules

- In bivalent sentential logic we had 9 rules: one for double negation (1), two for each other connective ($2 \times 4 = 8$)
- Here, each sentence can appear with a + or a - next to it!
- $\Rightarrow$ Twice as many rules: 18!
- You are *most definitely not* expected to memorise these

Tableaux for K3: double negation

$$\begin{array}{c} \sim\sim \varphi, + \\ | \\ \varphi, + \end{array} \qquad\qquad \begin{array}{c} \sim\sim \varphi, - \\ | \\ \varphi, - \end{array}$$

- Rationale: $\sim\sim \varphi$ and φ are also equivalent it K3 (same t. tables)

1	1
0	0
#	#

Tableaux for K3: conjunction

$$\begin{array}{c} \varphi\&\psi,+ \\ | \\ \varphi,+ \\ \psi,+ \end{array} \qquad\qquad \begin{array}{c} \varphi\&\psi,- \\ \diagup \quad \diagdown \\ \varphi,- \qquad \psi,- \end{array}$$

- Rationale:

&	1	0	#
1	1	0	#
0	0	0	0
#	#	0	#

Tableaux for K3: disjunction

$$\begin{array}{c} \varphi\vee\psi,+ \\ \diagup \quad \diagdown \\ \varphi,+ \qquad \psi,+ \end{array} \qquad\qquad \begin{array}{c} \varphi\vee\psi,- \\ | \\ \varphi,- \\ \psi,- \end{array}$$

- Rationale:

$\vee$	1	0	#
1	1	1	1
0	1	0	#
#	1	#	#

Tableaux for K3: negated conjunction

$$\begin{array}{c} \sim(\varphi\&\psi),+ \\ \diagup \quad \diagdown \\ \sim\varphi,+ \qquad \sim\psi,+ \end{array} \qquad\qquad \begin{array}{c} \sim(\varphi\&\psi),- \\ | \\ \sim\varphi,- \\ \sim\psi,- \end{array}$$

- Rationale: $\sim(\varphi\&\psi)$ and $\sim\varphi\vee\sim\psi$ are also equivalent it K3 (same t. tables)

	1	0	#
1	0	1	#
0	1	1	1
#	#	1	#

Tableaux for K3: negated disjunction

$$\begin{array}{c} \sim(\varphi\vee\psi),+ \\ | \\ \sim\varphi,+ \\ \sim\psi,+ \end{array} \qquad\qquad \begin{array}{c} \sim(\varphi\vee\psi)- \\ \diagup \quad \diagdown \\ \sim\varphi,- \qquad \sim\psi,- \end{array}$$

- Rationale: $\sim(\varphi\vee\psi)$ and $\sim\varphi\&\sim\psi$ are also equivalent it K3 (same t. tables)

	1	0	#
1	0	0	0
0	0	1	#
#	0	#	#

Tableaux for K3: conditional

$$\begin{array}{cc} \multicolumn{2}{c}{\varphi \supset \psi, +} \\ \sim \varphi, + & \psi, + \end{array} \qquad\qquad \begin{array}{c} \varphi \supset \psi, - \\ | \\ \sim \varphi, - \\ \psi, - \end{array}$$

- Rationale: $\varphi \supset \psi$ and $\sim \varphi \vee \psi$ are also equivalent it K3 (same t. tables)

	1	0	#
1	1	0	#
0	1	1	1
#	1	#	#

Tableaux for K3: negated conditional

$$\begin{array}{c} \sim (\varphi \supset \psi)+ \\ | \\ \varphi, + \\ \sim \psi, + \end{array} \qquad\qquad \begin{array}{cc} \multicolumn{2}{c}{\sim (\varphi \supset \psi), -} \\ \varphi, - & \sim \psi, - \end{array}$$

- Rationale: $\sim (\varphi \supset \psi)$ and $\varphi \& \sim \psi$ are also equivalent it K3 (same t. tables)

	1	0	#
1	0	1	#
0	0	0	0
#	0	#	#

Tableaux for K3: biconditional

$$\begin{array}{cc} \multicolumn{2}{c}{\varphi \equiv \psi, +} \\ \varphi, + & \sim \varphi, + \\ \psi, + & \sim \psi, + \end{array} \qquad\qquad \begin{array}{cccc} \multicolumn{4}{c}{\varphi \equiv \psi, -} \\ \sim \varphi, + & \varphi, + & \psi, - & \varphi, - \\ \psi, + & \sim \psi, + & \sim \psi, - & \sim \varphi, - \end{array}$$

- Rationale:

≡	1	0	#
1	1	0	#
0	0	1	#
#	#	#	#

Tableaux for K3: negated biconditional

$$\begin{array}{cc} \multicolumn{2}{c}{\sim (\varphi \equiv \psi), +} \\ \sim \varphi, + & \varphi, + \\ \psi, + & \sim \psi, + \end{array} \qquad\qquad \begin{array}{cccc} \multicolumn{4}{c}{\sim (\varphi \equiv \psi), -} \\ \varphi, + & \sim \varphi, + & \psi, - & \varphi, - \\ \psi, + & \sim \psi, + & \sim \psi, - & \sim \varphi, - \end{array}$$

- Rationale:

≡	1	0	#
1	0	1	#
0	1	0	#
#	#	#	#

Example 1: closed tableau

- We show that $p \supset q, q \supset r \vdash_{K3} p \supset r$

$$\checkmark\, p \supset q, +$$
$$\checkmark\, q \supset r, +$$
$$\checkmark\, p \supset r, -$$
$$|$$
$$\sim p, -$$
$$r, -$$

$$\begin{array}{ccc} \sim p, + & & q, + \\ \times & \sim q, + & r, + \\ & \times & \times \end{array}$$

(Branch b closes iff either (i) $\varphi, +$ and $\varphi, -$ on b or (ii) $\varphi, +$ and $\sim \varphi, +$ on b.)

Example 2: open tableau

- We show that $p \nvdash_{K3} q \vee \neg q$:

$$p, +$$
$$\checkmark\, q \vee \sim q, -$$
$$|$$
$$q, -$$
$$\sim q, -$$
$$\uparrow$$

Countermodel: $v(p) = 1$ and $v(q) = \#$

(If b open, then (i) if $\varphi, +$ is on b, then $v(\varphi) = 1$, (ii) if $\sim \varphi, +$ is on b, then $v(\varphi) = 0$, (iii) otherwise $v(\varphi) = \#$)

References

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