MODELING MODAL AGNOSTICISM

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Introduction

- Large literature on rational acceptability of *conditionals* wrt epistemic states modeled as orderings of worlds by comparative plausibility.
- Comparatively neglected topic: rational acceptability of *modals*, of the form 'It might/must be the case that *P*'. (One of Hansson's 'ten philosophical problems in belief revision'; Hansson [2003])
- Received view: Levi [1988], whose acceptance conditions impose strong constraints on rational agents.
- In a recent *Mind* article, Sorensen [2009] puts forward considerations that suggest that these constraints are *too* strong.

Introduction

- In this talk: discussion of impact of Sorensen's view on the standard world-order model of epistemic states.
- I offer a required generalization of the standard model.
- I also briefly discuss an associated modal logic with a clear supervaluationist flavour.

Outline

Languages

- $\mathcal{L}_0$ Arbitrary 'factual' propositional language, generated from a finite (!) set $\mathcal{A}$ of atomic sentences using the Boolean connectives $\{\wedge, \vee, \rightarrow, \neg\}$.
- W_0 Set of valuations of $\mathcal{L}_0$.
- $[\![\varphi]\!]$ Set of all $x \in W_0$, such that $x \models \varphi$.
- $\mathcal{L}_M$ Extension of $\mathcal{L}_0$, adding a unary possibility connective $\Diamond$.
- Intended interpretation:

  'It might be the case that...' (≠ 'It might *have been*...'!!)
  'There is a possibility that...' (≠ 'There *would have been*...'!!)
- $\Box$ Shorthand for $\neg \Diamond \neg$

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Consequence

- Cn Consequence operator:
  - Function from $\wp(\mathcal{L}_M)$ to $\wp(\mathcal{L}_M)$.
  - $\varphi \in \mathrm{Cn}(\Gamma)$ iff there exist $\varphi^* \in \mathcal{L}_0$ and $\Gamma^* \subseteq \mathcal{L}_0$ such that

    (i) φ and Γ can be obtained from φ^* and Γ^* by uniform substitution of sentences.
    (ii) φ^* is a classical consequence of Γ^*.
- $\Gamma \subseteq \mathcal{L}$ is consistent iff there exists $\varphi \in \mathcal{L}_M$, such that $\varphi \notin \mathrm{Cn}(\Gamma)$.

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Beliefs and epistemic states

- **E** Set of 'epistemic states' (more on these shortly).
- **B** Set of 'belief sets', subsets of $\mathcal{L}_M$ that have *at least* the following properties, for all $b \in \mathbf{B}$:
  - **Closure (Cl)** $\mathrm{Cn}(b) \subseteq b$.
  - **Consistency (Con)** b is consistent.
- Bel Belief function from **E** to **B**.
- Interpretation: gives us the beliefs that an agent is permitted to hold, given his or her epistemic state.

Orders

- A preorder $\geq$ on a set S is a binary relation on S that is both reflexive and transitive.
- $\sim$ The symmetric part of a preorder $\geq$.
- $\max(S, \geq)$ The set of maximal elements of S according to $\geq$, i.e. $\{x \in S : \forall x^* \in S,\ x \geq x^*\}$.
- W_{n+1} Set of all total preorders over W_n, where $n \in \mathbb{N}_0$.
- W Union of the W_i.

Levi's suggestion

- A pair of proposals regarding $\Diamond$ (Levi [1988]):
  - ($\Diamond$1) For all $\varphi \in \mathcal{L}_M$ and $x \in \mathbf{E}$, $\neg\Diamond\neg\varphi \in \text{Bel}(x)$ iff $\varphi \in \text{Bel}(x)$.
  - ($\Diamond$2) For all $\varphi \in \mathcal{L}_M$ and $x \in \mathbf{E}$, if $\varphi \notin \text{Bel}(x)$, then $\Diamond\neg\varphi \in \text{Bel}(x)$.
- ($\Diamond$1) seems clearly correct.
- Note in passing that, conveniently:

  Observation 1: Given (Cl) and (Con), ($\Diamond$1 $\Leftarrow$) entails that $\varphi \wedge \Diamond\neg\varphi \notin \text{Bel}(x)$, for all $\varphi \in \mathcal{L}_M$ and $x \in \mathbf{E}$. [Proof ▷]
- ($\Diamond$2), however, may be more problematic.

Two consequences of ($\Diamond$2)

- Trivially:

  Observation 2: ($\Diamond$1 $\Leftarrow$) and ($\Diamond$2) jointly entail (OM). [Proof ▷]

  Where:

  Opinionation wrt Modals (OM) For all $\varphi \in \mathcal{L}_M$ and $x \in \mathbf{E}$, either $\Diamond\varphi \in \text{Bel}(x)$ or $\neg\Diamond\varphi \in \text{Bel}(x)$.
- Relatedly:

  Observation 3: Given (Con), ($\Diamond$1 $\Leftarrow$) and ($\Diamond$2) jointly entail (Red). [Proof ▷]

  Where:

  Reduction (Red) For all $\varphi \in \mathcal{L}_M$ and $x \in \mathbf{E}$:

  (i) If $\Diamond\Diamond\varphi \in \text{Bel}(x)$ then $\Diamond\varphi \in \text{Bel}(x)$.
  (ii) If $\Diamond\Box\varphi \in \text{Bel}(x)$ then $\Box\varphi \in \text{Bel}(x)$.

Sorensen's objections

- Sorensen [2009]: (OM) and (Red) seem too strong.
- Regarding (Red):

  B: There might be a possibility of still getting that grant.
  A: There *is* a possibility that we'll still get the grant?
  B: That's not what I said: there *might* be such a possibility...

- Regarding (OM):

  A: Do you think that there's a possibility that we will get that grant?
  B: I don't know. Perhaps it's already too late.

Introducing ($\Diamond$3)

- If we grant that ($\Diamond$2) must go, we are quickly led to the following mild strengthening of its negation (argument omitted):

  If it is permissible to suspend judgment on φ, then it is optional to do so without accepting $\Diamond\neg\varphi$.

- More formally:

  ($\Diamond$3) For all $\varphi \in \mathcal{L}_M$, there exists $x \in \mathbf{E}$ such that $\varphi, \neg\varphi, \Diamond\varphi \notin \text{Bel}(x)$ iff there exists $y \in \mathbf{E}$ such that $\varphi, \neg\varphi \notin \text{Bel}(y)$ and $\Diamond\varphi \in \text{Bel}(y)$.

- Question: If ($\Diamond$3) is correct, what impact, if any, does this have on the standard view of epistemic states?

The standard view

- The view in question:

  Total Preorder in W_1 (TP$_1$) $\mathbf{E} = W_1$.
- Equally standardly:

  (Fac) For all $\varphi \in \mathcal{L}_0$ and $x \in \mathbf{E}$, $\varphi \in \text{Bel}(x)$ iff $\max(W_0, x) \subseteq [\![\varphi]\!]$.
- It is easy to establish, however, that:

  Observation 4: (TP$_1$), (Fac) and ($\Diamond$3) are jointly inconsistent.
  [Proof ▷]
- To keep ($\Diamond$3) we'll need to enlarge **E** by weakening (TP$_1$).

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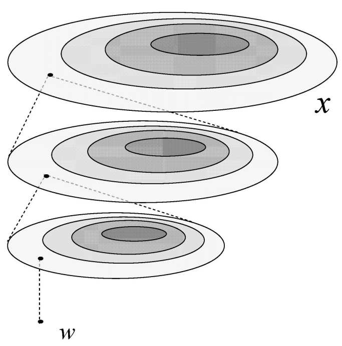
Meta-orderings

- One straightforward move that does the job:

  Total Preorder in $W - W_0$ (TP) $\mathbf{E} = W - W_0$.
- Illustration, where $x \in W_3$:



Defining Bel

- We then define Bel inductively.
- Basis step:

  (BS) For all x in W_1:

  (a) For all $\varphi \in \mathcal{L}_0$, $\varphi \in \text{Bel}(x)$ if, for all $y \in \max(W_0, x)$, $y \in [\![\varphi]\!]$.
  (b) For all $\varphi \in \mathcal{L}_M$, $\Box\varphi \in \text{Bel}(x)$, if $\varphi \in \text{Bel}(x)$.
  (c) For all $\varphi \in \mathcal{L}_M$, $\Diamond\varphi \in \text{Bel}(x)$ if $\varphi \in \text{Cn}(\{\psi, \chi\})$ for (i) some $\chi \in \text{Bel}(x)$ and (ii) some $\psi \in \mathcal{L}_0$ such that, for some $y \in \max(W_0, x)$, $y \in [\![\psi]\!]$.
  (d) For all $\Gamma \subseteq \mathcal{L}_M$, $\text{Cn}(\Gamma) \subseteq \text{Bel}(x)$ if $\Gamma \subseteq \text{Bel}(x)$.
  (e) For all $\varphi \in \mathcal{L}_M$, $\varphi \in \text{Bel}(x)$ only if its membership can be derived from (a)-(d).
- It can be proven that:

  Observation 5: Given (TP_1), (BS) entails ($\Diamond$1) and ($\Diamond$2). [Proof ▷]

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Defining Bel (ctd.)

- **$(\!|.|\!)$** For all $x \in \mathbf{E}$ and $\varphi \in \mathcal{L}_M$, $x \in (\!|\varphi|\!)$ iff $\varphi \in \text{Bel}(x)$.
- Inductive step:

  (IS) For all x in W_{n+1} $(n \geq 1)$:

  (a) For all $\varphi \in \mathcal{L}_M$, $\varphi \in \text{Bel}(x)$ if, for all $y \in \max(W_n, x)$, $y \in (\!|\varphi|\!)$.
  (b) For all $\varphi \in \mathcal{L}_M$, $\Box\varphi \in \text{Bel}(x)$, if $\varphi \in \text{Bel}(x)$.
  (c) For all $\varphi \in \mathcal{L}_M$, $\Diamond\varphi \in \text{Bel}(x)$ if $\varphi \in \text{Cn}(\{\psi, \chi\})$ for (i) some $\chi \in \text{Bel}(x)$ and (ii) some $\psi \in \mathcal{L}_M$ such that, for some $y \in \max(W_n, x)$, $y \in (\!|\psi|\!)$
  (d) For all $\Gamma \subseteq \mathcal{L}_M$, $\text{Cn}(\Gamma) \subseteq \text{Bel}(x)$ if $\Gamma \subseteq \text{Bel}(x)$.
  (e) For all $\varphi \in \mathcal{L}_M$, $\varphi \in \text{Bel}(x)$ only if its membership can be derived from (a)-(d).
- As promised:

  Observation 6: Given (TP), (BS) and (IS) jointly entail ($\Diamond$1) and ($\Diamond$3). [Proof ▷]

Logics

- We can use the two models presented here to define a 'consequence' relation on $\wp(\mathcal{L}_M) \times \mathcal{L}_M$.
- $\vDash_M$ Where $\Gamma \subseteq \mathcal{L}_M$ and $\varphi \in \mathcal{L}_M$, $\Gamma \vDash_M \varphi$ iff $\varphi \in \mathrm{Bel}(x)$ for all $x \in \mathbf{E}$, such that $\Gamma \subseteq \mathrm{Bel}(x)$.
- $\vDash_M$ looks very much like supervaluationist global consequence:

  Observation 7: Even given (TP_1), $\vDash_M$ fails to satisfy (i) contraposition, (ii) conditional proof and (iii) reasoning by cases. [Proof ▷]
- Furthermore:

  Observation 8: Given (TP_1), the S5 axioms are $\vDash_M$-valid. [Proof ▷]
- Question: What happens if we retreat to (TP)?

Concluding comments

- (TP) yields a very large set of epistemic states.
- Modal agnosticism can be similarly accommodated in models that are more quantitatively parsimonious (e.g. epist. states as sets of sets. . . of elements of W_1).
- However:
  - (a) Such models are arguably not as *qualitatively* parsimonious (orderings + sets vs orderings all the way up)
  - (b) (TP) turns out to have some interesting applications to the issue of *left-nested conditionals*.
- But (b) is another talk altogether. . .

Thank you!

Questions and comments welcome:
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References

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- Hansson, S.O. [2003]. Ten Philosophical Problems in Belief Revision. *Journal of Logic and Computation* 13, pp. 37–49.
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Proof of Observation 1

(1) For some $\varphi \in \mathcal{L}_M$ and $x \in \mathbf{E}$, $\varphi \wedge \Diamond\neg\varphi \in \mathrm{Bel}(x)$ [for reductio]

(2) $\Diamond\neg\varphi \in \mathrm{Bel}(x)$. [(1), (Cl)]

(3) If $\varphi \in \mathrm{Bel}(x)$, then $\Diamond\neg\varphi \notin \mathrm{Bel}(x)$ [($\Diamond$1 $\Leftarrow$), (Con)]

(4) $\varphi \notin \mathrm{Bel}(x)$ [(2), (3)]

(5) $\varphi \in \mathrm{Bel}(x)$ [(1), (Cl)] $\square$

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Proof of Observation 2

(1) For all $\varphi \in \mathcal{L}_M$ and $x \in \mathbf{E}$, either (i) $\neg\varphi \in \mathrm{Bel}(x)$ or (ii) $\neg\varphi \notin \mathrm{Bel}(x)$

(2) Assume (i)

(3) $\neg\Diamond\varphi \in \mathrm{Bel}(x)$ [($\Diamond$1 $\Leftarrow$)]

(4) Assume (ii)

(5) $\Diamond\varphi \in \mathrm{Bel}(x)$ [($\Diamond$2)] $\square$

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Proof of Observation 3

(1) For some $\varphi \in \mathcal{L}_M$ and $x \in \mathbf{E}$, $\Diamond\Diamond\varphi \in \text{Bel}(x)$ [for CP]

(2) $\neg\Diamond\neg\neg\Diamond\varphi \notin \text{Bel}(x)$ [(1), (Cl), (Con)]

(3) $\neg\Diamond\varphi \notin \text{Bel}(x)$ [(2), contrapositive of ($\Diamond$1 $\Leftarrow$)]

(4) $\Diamond\varphi \in \text{Bel}(x)$ [(3), ($\Diamond$1 $\Leftarrow$) and ($\Diamond$2) via (OM)] □

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Proof of Observation 4

(1) Let $\mathcal{A} = \{\varphi\}$, with $w, w^* \in W_0$, such that $w \vDash \varphi$ and $w^* \vDash \neg\varphi$

(2) There exists a *unique* $x \in \mathbf{E}$ such that $\varphi, \neg\varphi \notin \text{Bel}(x)$: the $x \in \mathbf{E}$, such that $w \sim_x w^*$ [(1), (TP_1), (Fac)]

(3) It isn't the case that $\Diamond\varphi \in \text{Bel}(x)$ and $\Diamond\varphi \notin \text{Bel}(x)$

(4) ($\Diamond$3) is false [(2), (3)] □

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Proof of Observation 5

($\Diamond$1): Given (TP_1), (BS)(b) is the only way to secure membership of $\mathrm{Bel}(x)$ for any $\Box\varphi$, such that $\varphi \in \mathcal{L}_M$, for any $x \in \mathbf{E}$. The desired conclusion then follows from (BS)(e).

($\Diamond$2): We first define a function $d : \mathcal{L}_M \mapsto \mathbb{N}_0$ as follows:

(i) For all $\varphi \in \mathcal{L}_0$, $d(\varphi) = 0$

(ii) For all $\varphi \in \mathcal{L}_M$, $d(\neg\varphi) = d(\varphi)$

(iii) For all $\varphi, \psi \in \mathcal{L}_M$, $d(\varphi \vee \psi) = d(\varphi \wedge \psi)$
$= d(\varphi \rightarrow \psi) = \max\{d(\varphi), d(\psi)\}$

(iv) For all $\varphi \in \mathcal{L}_M$, $d(\Diamond\varphi) = d(\varphi) + 1$

We now define:

$$\mathcal{L}_n := \{\varphi \in \mathcal{L}_M : d(\varphi) \leq n\}$$

Note that $\mathcal{L}_M = \bigcup \mathcal{L}_n$, $n \in \mathbb{N}_0$.

Proof of Observation 5 (ctd.)

We also define the corresponding restricted version of ($\Diamond$2):

($\mathcal{L}_n \Diamond 2$) For all $\varphi \in \mathcal{L}_n$ and all $x \in \mathbf{E}$, if $\varphi \notin \mathrm{Bel}(x)$, then $\Diamond\neg\varphi \in \mathrm{Bel}(x)$.

We first prove ($\mathcal{L}_0 \Diamond 2$). Assume for CP that $\varphi \notin \mathrm{Bel}(x)$, where $\varphi \in \mathcal{L}_0$. It follows, given the contrapositive of (BS)(a), that, for some $y \in \max(W_0, x)$, $y \in [\![\neg\varphi]\!]$. From this, given (BS)(c), we recover the fact that $\Diamond\neg\varphi \in \mathrm{Bel}(x)$.

We now prove that if ($\mathcal{L}_n \Diamond 2$), then ($\mathcal{L}_{n+1} \Diamond 2$), where $n \in \mathbb{N}_0$. Consider an arbitrary $\varphi \in \mathcal{L}_{n+1}$. Let $\mathrm{DNF}(\varphi)$ denote its DNF, i.e. its equivalent disjunction of conjunctions of sentences ψ_i, such that (i) ψ_i is a literal or (ii) $\psi_i = \Diamond\chi$ or $\psi_i = \neg\Diamond\chi$, where $\chi \in \mathcal{L}_n$:

$$\mathrm{DNF}(\varphi) = (\psi_1, \wedge \ldots) \vee (\psi_n, \wedge \ldots) \vee \ldots$$

Proof of Observation 5 (ctd.)

Call a sentence 'indefinite' iff neither it nor its negation is in $\text{Bel}(x)$.

$\varphi \notin \text{Bel}(x)$ iff either (a) φ is definite and $\neg\varphi \in \text{Bel}(x)$ or (b) φ is indefinite.

If (a), it immediately follows, by (IS)(c), that $\Diamond\neg\varphi \in \text{Bel}(x)$.

Assume ($\mathcal{L}_n \Diamond 2$). It follows from that, alongside ($\Diamond 1$), that, for all $\chi \in \mathcal{L}_n$ and $x \in \mathbf{E}$, either $\Diamond\chi \in \text{Bel}(x)$ or $\neg\Diamond\chi \in \text{Bel}(x)$ (see proof of Obs 2).

In other words: all non-literal conjuncts in $\text{DNF}(\varphi)$ are definite.

Proof of Observation 5 (ctd.)

So if (b), then $\text{DNF}(\varphi)$ must (i) include at least one indefinite disjunct, which itself, by the previous remark, must contain at least one indefinite *literal* conjunct, and (ii) only include definite disjuncts whose negation is in $\text{Bel}(x)$.

Now assume (b) and consider $\neg\text{DNF}(\varphi)$. This will be equivalent to a conjunction α of disjunctions that either (i) contain an indefinite literal disjunct, namely the negation of the corresponding conjunct in $\text{DNF}(\varphi)$, or (ii) are members of $\text{Bel}(x)$:

$$\alpha = (\neg\psi_1, \vee \ldots) \wedge (\neg\psi_n, \vee \ldots) \wedge \ldots$$

Proof of Observation 5 (ctd.)

Let Γ denote the set of indefinite literals in α. Since $\text{DNF}(\varphi) \notin \text{Bel}(x))$, it follows that the disjunction of their negations isn't in $\text{Bel}(x)$. It then follows that there exists $y \in \max(W_0, x)$ such that $y \in [\![\bigwedge \Gamma]\!]$.

Now it is easy to show that α is a joint consequence of $\bigwedge \Gamma$ and the conjunction of the conjuncts in α that are members of $\text{Bel}(x)$ (if any).

It then follows from (BS)(c) that $\Diamond\alpha \in \text{Bel}(x)$ and hence, by (BS)(d), that $\Diamond\neg\varphi \in \text{Bel}(x)$.

We can therefore conclude that $(\mathcal{L}_{n+1} \Diamond 2)$ holds. □

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Proof of Observation 6

($\Diamond$1): Call $(W_n \Diamond 1)$, the restriction of ($\Diamond$1) to W_n. We saw in Observation 5 that (BS) entails that $(W_1 \Diamond 1)$ holds, given (BS)(e) and the fact that (BS)(b) is the only way to secure membership of $\Box\varphi$ when $x \in W_1$. For similar reasons, (IS) entails that $(W_n \Diamond 1)$, for $n \geq 2$.

($\Diamond$3 $\Leftarrow$): We prove this for $x \in W_n$, where $n \geq 2$; the proof for $n = 1$ is analogous (simply swap $[\![\,.\,]\!]$ for $(\!|\,.\,|\!)$).

Assume for CP that there exists $y \in W_n$ such that $\varphi, \neg\varphi \notin \text{Bel}(y)$ and $\Diamond\varphi \in \text{Bel}(y)$.

It follows from (IS)(a) and the fact that $\varphi \notin \text{Bel}(y)$, that there exists $x \in W_{n-1}$ such that $x \in \max(W_{n-1})$ and $x \notin (\!|\varphi|\!)$.

Proof of Observation 6 (ctd.)

There exists $y^* \in W_n$ such that $\max(W_{n-1}, y^*) = \{x\}$, as well as $z \in W_{n+1}$ such that $\max(W_n, z) = \{y, y^*\}$.

Since, as is easily verified, $\Diamond\varphi \notin \text{Bel}(y^*)$ and since $\varphi, \neg\varphi \notin \text{Bel}(y)$, it follows that $\varphi, \neg\varphi, \Diamond\varphi \notin \text{Bel}(z)$.

($\Diamond$3 $\Rightarrow$): Assume for CP that there exists $y \in W_n$ such that $\varphi, \neg\varphi, \Diamond\varphi \notin \text{Bel}(y)$.

From the fact that $\neg\varphi \notin \text{Bel}(y)$, it follows that, for some $x \in W_1$, $\neg\varphi \notin \text{Bel}(x)$. Indeed, assume for reductio, that there is no such x. It then follows that $\neg\varphi \in \text{Bel}(y)$, contrary to our initial assumption, since if for all $x \in W_n$, $\neg\varphi \in \text{Bel}(x)$, then, trivially, for all $x^* \in W_{n+1}$, for all $x \in \max(W_n, x^*)$, $x \in (\!|\neg\varphi|\!)$ and hence by (IS)(a), $\neg\varphi \in \text{Bel}(x^*)$.

Proof of Observation 6 (ctd.)

Either (i) $\varphi \in \text{Bel}(x)$ or (ii) $\varphi \notin \text{Bel}(x)$.

Assume (i). Now:

> **Observation 9:** If there exists $x \in W_n$ $(n \geq 1)$ such that $\varphi \in \text{Bel}(x)$, then there exists $x^* \in W_{n+1}$ such that $\varphi \in \text{Bel}(x^*)$.

Indeed, consider any x^* such that $\max(W_n, x^*) = \{x\}$.

So, by Obs 9, there exists $y^* \in W_n$ such that $\varphi \in \text{Bel}(y^*)$. There also exists $z \in W_{n+1}$, such that $\max(W_n, z) = \{y, y^*\}$.

Since $\varphi \in \text{Bel}(y^*)$, by (IS)(c), $\Diamond\varphi \in \text{Bel}(z)$. Since $\varphi, \neg\varphi \notin \text{Bel}(y)$, by (IS), $\varphi, \neg\varphi \notin \text{Bel}(z)$.

Assume (ii). By ($\Diamond$2), which holds for all $x \in W_1$ (see Observation 5), since $\neg\varphi \notin \text{Bel}(x)$, $\Diamond\varphi \in \text{Bel}(x)$. □

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Proof of Observation 7

(i) For all $\varphi \in \mathcal{L}_M$, $\varphi \vDash_M \Box\varphi$. However, it is not the case that for all $\varphi \in \mathcal{L}_M$, $\Diamond\neg\varphi \vDash_M \neg\varphi$. Countermodel: see epistemic state x in proof of Observation 4.

(ii) For all $\varphi \in \mathcal{L}_M$, $\varphi \vDash_M \Box\varphi$. However, it is not the case that for all $\varphi \in \mathcal{L}_M$, $\vDash_M \varphi \to \Box\varphi$. Countermodel: same as above.

(iii) For all $\varphi \in \mathcal{L}_M$, $\varphi \vDash_M \Box\varphi \vee \Box\neg\varphi$ and $\neg\varphi \vDash_M \Box\varphi \vee \Box\neg\varphi$. However, it is not the case that for all $\varphi \in \mathcal{L}_M$, $\varphi \vee \neg\varphi \vDash_M \Box\varphi \vee \Box\neg\varphi$. Countermodel: same as above. □

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Proof of Observation 8

$\vDash_M$ **K:** Either (i) $\Diamond\neg\varphi \in \mathrm{Bel}(x)$ or (ii) $\Diamond\neg\varphi \notin \mathrm{Bel}(x)$. Furthermore, either (iii) $\neg\Diamond\neg\psi \in \mathrm{Bel}(x)$ or (iv) $\neg\Diamond\neg\psi \notin \mathrm{Bel}(x)$.

Now from (TP_1) and (BS), we recover (◇1) and (◇2) (see Obs. 5). From these, we obtain (OM) (see Obs. 2).

If (iv) then, by (OM), $\Diamond\neg\psi \in \mathrm{Bel}(x)$. If, furthermore, (ii) then, again by (OM), it follows that $\neg\Diamond\neg\varphi \in \mathrm{Bel}(x)$ and therefore, by (BS), that $\varphi \in \mathrm{Bel}(x)$. Finally, by (BS)(c), $\Diamond(\varphi \wedge \neg\psi) \in \mathrm{Bel}(x)$.

If (i), or (ii), then $\Diamond\neg\varphi \vee \neg\Diamond\neg\psi \in \mathrm{Bel}(x)$

So, whatever way, $\Diamond(\varphi \wedge \neg\psi) \vee (\Diamond\neg\varphi \vee \neg\Diamond\neg\psi) \in \mathrm{Bel}(x)$, and hence, by (BS)(d), $\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi) \in \mathrm{Bel}(x)$.

Proof of Observation 8 (ctd.)

$\vDash_M$ **T:** By ($\Diamond$2), either $\varphi \in \mathrm{Bel}(x)$ or $\Diamond\neg\varphi \in \mathrm{Bel}(x)$ and hence, by (BS)(d) $\Box\varphi \to \varphi \in \mathrm{Bel}(x)$.

$\vDash_M$ **5:** By (OM), either $\Diamond\varphi \in \mathrm{Bel}(x)$ or $\neg\Diamond\varphi \in \mathrm{Bel}(x)$. By (BS)(b) and (IS)(b), if $\Diamond\varphi \in \mathrm{Bel}(x)$, then $\Box\Diamond\varphi \in \mathrm{Bel}(x)$. Therefore, by (BS)(d), $\Diamond\varphi \to \Box\Diamond\varphi \in \mathrm{Bel}(x)$. $\Box$

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