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[2] A Little More Set Theory

0. Outline

1. Set theory basics (ctd.)
2. Operations on sets
3. Relations
4. Probability theory basics

1. Set theory basics (ctd.)

- *Power sets*:
  - The set of all subsets of a set S (denoted $\wp(S)$ or 2^S) is known as the *power set* of S.
  - Note that $|\wp(S)| = 2^{|S|}$ (this is because each subset S^* of S corresponds to a choice, for each element of S, as to whether of not that element belongs to S^* - that's 2 options for each element of S)
  - Example: let $S = \{a,b,c\}$, then $\wp(S) = \{\emptyset,\{a\},\{b\},\{c\},\{a,b\},\{a,c\},\{b,c\}, S\}$

2. Operations on sets

- *Operations on sets*:
  - *Union*: $A \cup B =_{\text{def}} \{x \mid x \in A \text{ or } x \in B\}$

    Example: $\{a,b,c\} \cup \{c,d\} = \{a,b,c,d\}$
  - *Intersection*: $A \cap B =_{\text{def}} \{x \mid x \in A \text{ and } x \in B\}$

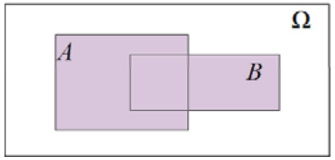
    Example: $\{a,b,c\} \cap \{c,d\} = \{c\}$
  - *Difference*: $A - B =_{\text{def}} \{x \mid x \in A \text{ and } x \notin B\}$

    Example: $\{a,b,c\} - \{c,d\} = \{a,b\}$

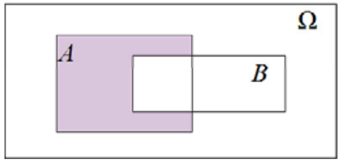
    Note: a special case of difference is *complementation*. The complement of A (A' or $\overline{A}$) is $\Omega - A$.
  - Note: the union and intersection of n sets A_1 to A_n are sometimes respectively denoted $\bigcup_{i=1}^{n}\{A_i\}$ and $\bigcap_{i=1}^{n}\{A_i\}$.

2. Operations on sets

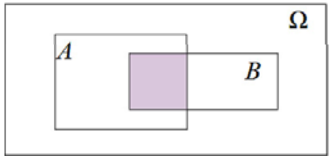
- $A \cup B$



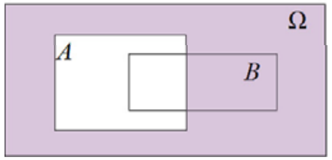
- $A - B$



- $A \cap B$



- $\overline{A}$

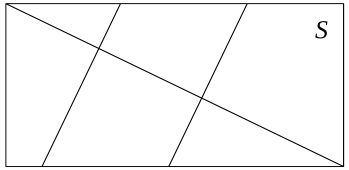


2. Operations on sets

- With the notions of union, intersection and complementation these notions in hand, we can define a *partition P* of a non-empty set *A*:

  Set *P* is a partition of non-empty set $A =_{def} P$ is a set of non-empty subsets of *A* such that (i) for any sets $X, Y \in S$, $X \cap Y = \varnothing$ and (ii) the union of the members of $P = S$.

  The members of a partition are known as '*cells*'.



2. Operations on sets

- Some *laws* (commonsense! - easy to verify on a Venn diagram):
  - *Indempotent laws:*

    (i) $A \cup A = A$ (ii) $A \cap A = A$
  - *Commutative laws:*

    (i) $A \cup B = B \cup A$ (ii) $A \cap B = B \cap A$
  - *Associative laws:*

    (i) $(A \cup B) \cup C = A \cup (B \cup C)$

    (ii) $(A \cap B) \cap C = A \cap (B \cap C)$
  - *Distributive laws:*

    (i) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

    (ii) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

2. Operations on sets

- And some more…
  - *Identity laws:*

    (i) $A \cup \varnothing = A$ (ii) $A \cap \varnothing = \varnothing$

    (iii) $A \cup \Omega = \Omega$ (iv) $A \cap \Omega = A$
  - *Complement laws:*

    (i) $A \cup \overline{A} = \Omega$ (ii) $A \cap \overline{A} = \varnothing$

    (iii) $\overline{\overline{A}} = A$ (iv) $A - B = A \cap \overline{B}$
  - *De Morgan's laws:*

    (i) $\overline{A \cup B} = \overline{A} \cap \overline{B}$ (ii) $\overline{A \cap B} = \overline{A} \cup \overline{B}$
  - *Consistency principle:*

    (i) $A \subseteq B$ iff $A \cup B = B$ (ii) $A \subseteq B$ iff $A \cap B = A$

3. Relations

- *Ordered pairs*: an ordered pair <*a*,*b*> is like a set of two elements in which order matters, so that whilst {*a*,*b*} = {*b*,*a*}, it generally isn't the case that <*a*,*b*> = <*b*,*a*>, unless *a* = *b*.

  Ordered pairs can be represented in set-theoretic terms: $<a,b> =_{def} \{\{a\}.\{a,b\}\}$
- *Cartesian products:* the Cartesian product of two sets *A* and *B*, denoted by $A \times B$, is the set of all ordered pairs in which the first element is taken from *A* and the second from *B*. In other words:

  $$A \times B = \{<x,y> \mid x \in A \text{ and } y \in B\}$$

  Note: the rationale for the term 'product' lies in the fact that $|A \times B| = |A| \times |B|$.

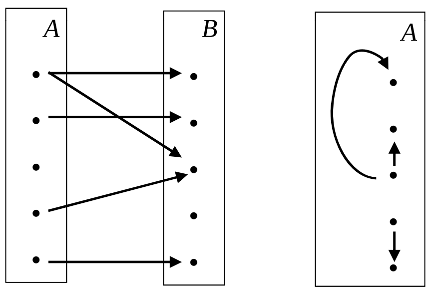
  Example: $\{a,b\} \times \{c,d\} = \{<a,c>,<a,d>,<b,c>,<b,d>\}$

3. Relations

- Binary relations:
  - a *binary relation R* between two sets *A* and *B* is a subset (proper or not) of $A \times B$ ($R \subseteq A \times B$).
  - when *R* holds between a set *A* and itself, it is known as a relation *in* or *on A*.
  - $<a,b> \in R$ is often written *aRb* or *Rab*.
- n-ary relations:
  - An *n-ary relation R* between *n* sets A_1 to A_n, is a subset (proper or not) of $A_1 \times \ldots \times A_n$ ($R \subseteq A_1 \times \ldots \times A_n$).

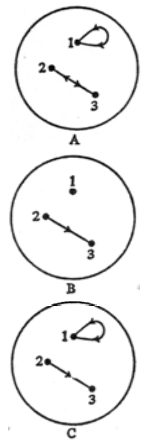
3. Relations

- Binary relations between *A* and *B* and on *A*, respectively (an arrow between points *x* and *y* indicates that $<x,y> \in R$):



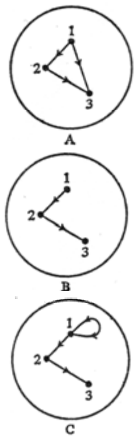
3. Relations

- Some important possible properties of binary relations on a set:
  - Symmetry
    - R is *symmetric* iff, for every $<x,y> \in R$, $<y,x> \in R$. [A; e.g. being in the same room as]
    - R is *asymmetric* iff, for every $<x,y> \in R$, $<y,x> \notin R$. [B; e.g. being a parent of]
    - R is *nonsymmetric* iff, for some $<x,y> \in R$, $<y,x> \notin R$. [C; e.g. being proud of]



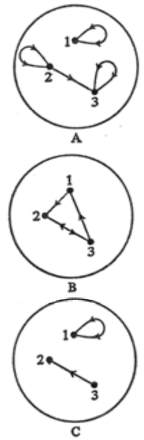
3. Relations

- Transitivity
  - *R* is *transitive* iff for every *x*, *y*, and *z* $\in A$, if $\langle x,y\rangle \in R$ and $\langle y,z\rangle \in R$, then $\langle x,z\rangle \in R$. [A; e.g. being taller than]
  - *R* is *intransitive* iff for every *x*, *y*, and $z \in A$, if $\langle x,y\rangle \in R$ and $\langle y,z\rangle \in R$, then $\langle x,z\rangle \notin R$. [B; e.g. being the mother of]
  - R is *nontransitive* iff for some *x*, *y*, and $z \in A$, $\langle x,y\rangle \in R$ and $\langle y,z\rangle \in R$, but $\langle x,z\rangle \notin R$. [C; e.g. being proud of]



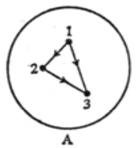
3. Relations

- Reflexivity
  - *R* is *reflexive* iff, for every $x \in A$, $\langle x,x\rangle \in R$. [A; e.g. being as tall as]
  - *R* is *irreflexive* iff, for every $x \in A$, $\langle x,x\rangle \notin R$. [B; e.g. being spied on by]
  - *R* is *nonreflexive* iff, for some $x \in A$, $\langle x,x\rangle \notin R$. [C; being proud of]



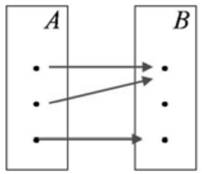
3. Relations

- Connectedness (aka 'completeness')
  - R is *connected* iff for every x and $y \in A$, $\langle x,y\rangle \in R$ or $\langle y,x\rangle \in R$. [A; e.g. being at least as heavy as]

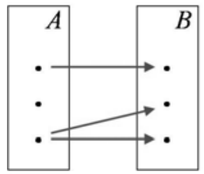


3. Relations

- *Functions*: a function (or 'map') *F* from a set *A* to a set *B* (denoted by '$F: A \mapsto B$') is a relation between *A* and *B* such that every member of *A* occurs as the first element of one and only one ordered pair in *F*.
- A function from *A* to *B*:



- *Not* a function from *A* to *B*:



4. Probability theory basics

- In L1, I briefly mentioned that Bayesians are committed, *inter alia*, to the following claim:

  PROB: If *S* is rational at time *t*, $Bel_{S,t}$ (the function that maps propositions onto degrees of belief of *S* at *t*) is a probability function.
- So what exactly *is* a probability function?*

  *Note: *two* definitions of 'pr. function' in the Bayesian literature.
  - A *stipulative definition* of 'probability function' as simply a kind of mathematical function with certain formal properties.
  - A (plausible but contested) *philosophical analysis* of the pretheoretic concept of 'probability function' as rational belief function. ('probability of *P*' means 'rat. deg. of belief in *P*')

4. Probability theory basics

- Obviously, on the second reading, PROB comes out as rather uninformative.
- In the next few slides, I will be using the term 'probability function' as a purely technical, stipulatively defined mathematical concept.

Next lecture: 'Probability Theory'

- Reading:
  - Weisberg, J. [unpublished] 'A Probability Primer for Philosophers', section 3 'Conditional probability'.