On Strengthening the Logic of Iterated Belief Revision: Proper Ordinal Interval Operators*

Richard Booth

Cardiff University

Jake Chandler

La Trobe University

Abstract

Darwiche and Pearl's seminal 1997 article outlined a number of baseline principles for a logic of iterated belief revision. These principles, the DP postulates, have been supplemented in a number of alternative ways. However, most of the suggestions for doing so have been radical enough to result in a dubious 'reductionist' principle that identifies belief states with orderings of worlds. The present paper offers a more modest strengthening of Darwiche and Pearl's proposal. While the DP postulates constrain the relation between a prior and a posterior conditional belief set, our new principles govern the relation between two posterior conditional belief sets obtained from a common prior by different revisions. We show that operators from the family that these principles characterise, which subsumes both lexicographic and restrained revision, can be represented as relating belief states that are associated with a 'proper ordinal interval' assignment, a structure more fine-grained than a simple ordering of worlds. We close the paper by noting that these operators satisfy iterated versions of a large number of AGM era postulates.

Keywords: AGM, belief revision, iterated belief change, iterated revision

1. Introduction

Belief change is a well established subfield of AI, with important connections to a number of other areas, most notably nonmonotonic reasoning (see for instance [2] for an early discussion). One of the most pressing and contentious issues that it faces is the handling of so-called 'iterated' belief revision: establishing plausible rationality constraints on the result of a sequence of successive revisions. This was identified as an open problem in Hansson's 'Ten philosophical problems of belief revision' [3] and remains as such, over fifteen years later.

Darwiche & Pearl's [4] seminal paper on the topic proposed a number of popular baseline principles for iterated revision, constraining the relation between the beliefs resulting from single revisions (that is, the agent's prior 'conditional beliefs') and the beliefs resulting from sequences of two revisions (equivalently: the agent's posterior conditional beliefs after a single revision).

These principles, the DP postulates, have been strengthened in various manners. However, most proposals for doing so–such as natural [5], lexicographic [6], and restrained [7] revision (see [8] for an overview)–have yielded sets of principles that are powerful enough to entail the following strong 'reductionist' principle: the set of beliefs held by an agent after a sequence of two revisions is fully determined by the agent's single-step revision dispositions or again conditional beliefs. Given the AGM postulates for single-step revision of Alchourrón, Gärdenfors and Makinson [9], this thesis can alternatively be cashed out in terms of an identification of belief states, the relata of the revision function, with total preorders (TPO's) over possible worlds. Booth & Chandler [10] have recently provided a counterexample that shows the reductionist position to be too strong: states require more structure than that provided by a mere TPO.

In this paper, we supplement the DP postulates, alongside a popular principle termed '(P)' by Booth & Meyer [7] and 'Independence' by Jin & Thielscher [11], with a number of novel conditions, which, like these, only govern sequences of *two* revisions, rather than sequences of arbitrarily many such operations. But

2

while the DP postulates constrain the relation between a prior and a posterior conditional belief set, our new principles notably govern the relation between two posterior conditional belief sets obtained from a common prior by different revisions.

We take as our foil two postulates of this variety considered by Booth & Meyer [12]. These played a key role in characterising a family of so-called *non-prioritised* revision operators, for which they offered a representation in terms of what we shall call 'proper ordinal interval (POI) assignments'. We show that these two postulates become implausible in the context of *prioritised* revision, which is the focus of the present paper. First of all, they turn out to characterise lexicographic revision when one supplements the remaining postulates of Booth & Meyer, i.e. (P) and the DP postulates, with the AGM postulate of Success. Secondly, they fall prey to an intuitive class of counterexample.

If adding Success to Booth & Meyer's postulates is not an option, how might we devise a prioritised revision operator that satisfies the former while retaining as much of the latter as possible? Our strategy here proceeds semantically: we seek to minimally transform the posterior TPO obtained by Booth & Meyer's method of non-prioritised revision in such a way that Success is assured. We show that this is achieved by adding a 'naturalisation' step, which is essentially an application of Boutillier's natural revision operation.

We call the resulting family of iterated revision operators, which subsumes both lexicographic and restrained revision operators, the family of POI revision operators. These operators turn out to satisfy two weakenings of Booth & Meyer's postulates, which have not yet been discussed in the literature and which neatly avoid the counterexamples raised against their stronger counterparts. After characterising POI revision, both semantically and syntactically, we show how the additional structure of POI assignments also provides sufficient resources to handle Booth & Chandler's counterexample to reductionism. We close the paper by noting that POI revision operators satisfy iterated versions of a large number of AGM era postulates, including Superexpansion.

The plan of the remainder of the article is as follows. In the fairly substantial

Section 2, we first introduce some basic terminology and definitions, recapitulating some recent work on iterated revision which the remainder of the paper builds on. Section 3 introduces the two strong postulates of Booth & Meyer, whose associated semantic framework we then present in Section 4. These postulates are critically discussed in Section 5. Section 6 outlines our construction of the POI family of operators. In Section 7, we discuss the weakenings of Booth & Meyer's postulates that are satisfied by the members of our new family. In Section 8, the family is characterised semantically and syntactically, in two different manners. In Section 9, we show how the additional expressive power of POI assignments allows us to adequately model Booth & Chandler's counterexample to the equation of states with TPO's. We wrap up the paper with a discussion, in Section 10, of the extent to which the members of the POI family satisfy extensions of various strong AGM era postulates to the iterated case. We then conclude in Section 11. We provide a sizeable technical appendix, which contains an example of a family of POI revision operators with a 'concrete' representation of belief states (Appendix A), as well as complete proofs of the various technical results (Appendix B).

2. Preliminaries

The beliefs of an agent are represented by a *belief state* Ψ. Ψ determines a *belief set* $[\Psi]$, a deductively closed set of sentences, drawn from a finitely generated propositional, truth-functional language L. Logical equivalence is denoted by $\equiv$ and the set of logical consequences of $\Gamma \subseteq L$ by $\mathrm{Cn}(\Gamma)$. The set of propositional worlds is denoted by W, and the set of models of a given sentence A is denoted by $[\![A]\!]$. We occasionally abuse notation and use x to denote not a world but a sentence. In particular, whenever a world x appears within the scope of a logical connective, it should be understood as referring to some sentence whose set of models is exactly $\{x\}$. So, for example, given $x, y \in W$, the sentence $x \vee y$ is such that $[\![x \vee y]\!] = \{x, y\}$.

In terms of belief dynamics, our principal focus is on iterated revision–rather

than contraction–operators, which return, for any prior belief state Ψ and consistent sentence A, the posterior belief state $\Psi * A$ that results from an adjustment of Ψ to accommodate the inclusion of A in $[\Psi]$.

2.1. Single step revision

The function $*$ is assumed to satisfy the AGM postulates of [9, 4]–henceforth 'AGM', for short:

(K1*) $\mathrm{Cn}([\Psi * A]) \subseteq [\Psi * A]$

(K2*) $A \in [\Psi * A]$ (aka 'Success')

(K3*) $[\Psi * A] \subseteq \mathrm{Cn}([\Psi] \cup \{A\})$

(K4*)   If $\neg A \notin [\Psi]$, then $\mathrm{Cn}([\Psi] \cup \{A\}) \subseteq [\Psi * A]$

(K5*) $[\Psi * A]$ is consistent

(K6*)   If $A \equiv B$, then $[\Psi * A] = [\Psi * B]$

(K7*) $[\Psi * A \wedge B] \subseteq \mathrm{Cn}([\Psi * A] \cup \{B\})$

(K8*)   If $\neg B \notin [\Psi * A]$, then $\mathrm{Cn}([\Psi * A] \cup \{B\}) \subseteq [\Psi * A \wedge B]$

This ensures the following convenient representability of single-shot revision: each Ψ has associated with it a total preorder[2] $\preccurlyeq_\Psi$ over W such that $[\![\Psi * A]\!] = \min(\preccurlyeq_\Psi, [\![A]\!])$, where $\min(\preccurlyeq_\Psi, [\![A]\!]) := \{x \in [\![A]\!] \mid \forall y \in [\![A]\!], x \preccurlyeq y\}$ [13, 14]. This ordering is sometimes interpreted in terms of relative '(im)plausibility', so that $x \preccurlyeq_\Psi y$ iff x is considered at least as 'plausible' as y in state Ψ. In this context, Success corresponds to the requirement that $\min(\preccurlyeq_{\Psi * A}, W) \subseteq [\![A]\!]$.

The single-shot revision dispositions associated with Ψ can also be represented by a '*conditional belief set*' $[\Psi]_c$. This set extends the belief set $[\Psi]$ by further including various 'conditional beliefs', of the form $A \Rightarrow B$, where $\Rightarrow$ is a non-truth-functional conditional connective. This is achieved by means of the so-called Ramsey Test ([15], [16]), according to which $A \Rightarrow B \in [\Psi]_c$ iff $B \in [\Psi * A]$.

---

[2]A total preorder is a binary relation that is reflexive, transitive and complete.

Following convention, we shall call principles couched in terms of belief sets 'syntactic', and principles couched in terms of TPO's 'semantic'. The principles that we will discuss will be given in both types of format, with the distinction reflected in the nomenclature by the use of a subscript '$\preccurlyeq$' to denote semantic principles.

We shall also be touching on a broader class of *non-prioritised* iterated 're-vision' operators, for which Success does not necessarily hold. These will be denoted by the symbol $\circ$. To avoid ambiguity, we will follow a convention of superscripting every principle governing a belief change operator with the relevant operator symbol (here: $*$ or $\circ$).

2.2. Two step revision

In terms of its behaviour under two iterations, $*$ will be assumed to satisfy the DP postulates of [4], which constrain the belief set resulting from two successive revisions, or, equivalently, the conditional belief set resulting from a single revision:

(C1*)   If $A \in \mathrm{Cn}(B)$, then $[(\Psi * A) * B] = [\Psi * B]$

(C2*)   If $\neg A \in \mathrm{Cn}(B)$, then $[(\Psi * A) * B] = [\Psi * B]$

(C3*)   If $A \in [\Psi * B]$, then $A \in [(\Psi * A) * B]$

(C4*)   If $\neg A \notin [\Psi * B]$, then $\neg A \notin [(\Psi * A) * B]$

The semantic counterparts of these principles are given by:

(C1$^*_{\preccurlyeq}$)   If $x, y \in [\![A]\!]$, then $x \preccurlyeq_{\Psi * A} y$ iff $x \preccurlyeq_{\Psi} y$

(C2$^*_{\preccurlyeq}$)   If $x, y \in [\![\neg A]\!]$, then $x \preccurlyeq_{\Psi * A} y$ iff $x \preccurlyeq_{\Psi} y$

(C3$^*_{\preccurlyeq}$)   If $x \in [\![A]\!]$, $y \in [\![\neg A]\!]$ and $x \prec_{\Psi} y$, then $x \prec_{\Psi * A} y$

(C4$^*_{\preccurlyeq}$)   If $x \in [\![A]\!]$, $y \in [\![\neg A]\!]$ and $x \preccurlyeq_{\Psi} y$, then $x \preccurlyeq_{\Psi * A} y$

Booth & Chandler [10] have effectively recently shown that the DP postulates can be collectively recast in terms of a pair of binary relations of *overruling* and

strict overruling, introduced in [12]. These make a number of appearances later on in this paper and are defined as follows:

Definition 1. *Where A and B are consistent sentences in L, B overrules A (in Ψ) if $A \notin [(\Psi * A) * B]$, while B strictly overrules A (in Ψ) if $\neg A \in [(\Psi * A) * B]$.*[3]

Indeed, their Observation 2, building on a result of Booth & Meyer [12, Proposition 6], establishes, given the above definition, that the DP postulates are collectively equivalent to:

$$[(\Psi * A) * B] = \begin{cases} [\Psi * B], & \text{if } B \text{ strictly overrules } A \\ [\Psi * B] \cap [\Psi * A \wedge B], & \text{if } B \text{ overrules } A, \text{ but not strictly} \\ [\Psi * A \wedge B], & \text{if } B \text{ does not overrule } A \end{cases}$$

So, once the relation between B and A in terms of overruling is determined, the results of various *single* revisions by combinations of A and B are sufficient to give us the result of a sequence of *two* successive revisions by A and then B. To put this in the form of a slogan: Two-step revision is single-step revision plus overruling (or lack thereof).

Beyond the DP postulates, we impose two further constraints on two step revision. First, we impose a principle of irrelevance of syntax that we shall call 'Equivalence':

(Eq*)   If $A \equiv B$ and $C \equiv D$, then $[(\Psi * A) * C] = [(\Psi * B) * D]$

or semantically

(Eq$^*_\preccurlyeq$)   If $A \equiv B$, then $\preccurlyeq_{\Psi * A} = \preccurlyeq_{\Psi * B}$

---

[3]Incidentally, the first relation also corresponds to the condition under which Chandler [17] proposed that one takes B to provide a reason to not believe A. The second relation is related to the condition under which he claimed one takes B to provide a reason to believe $\neg A$ [18]. There is also a clear connection here with Pollock's well known concepts of undercutting and rebutting defeaters [19].

Second, we assume that $*$ satisfies the principle (P*), which strengthens both (C3*) and (C4*):

(P*)   If $\neg A \notin [\Psi * B]$, then $A \in [(\Psi * A) * B]$

Its semantic counterpart is given by:

(P$^*_\preccurlyeq$)   If $x \in [\![A]\!]$, $y \in [\![\neg A]\!]$ and $x \preccurlyeq_\Psi y$, then $x \prec_{\Psi * A} y$

Satisfaction of AGM, (Eq*), (C1*), (C2*) and (P*) means that $*$ is an '*admissible*' revision operator, in the sense of [7].

2.3. Reductionism: states as TPO's?

The constraints considered so far are notably satisfied by two well-known kinds of revision operators: restrained operators and lexicographic operators.[4] In semantic terms, these both promote the minimal A-worlds in the prior TPO to become minimal worlds in the posterior TPO. Regarding the rest of the ordering, restrained revision operators preserve the strict ordering $\prec_\Psi$ while additionally making every A-world x strictly lower ranked than every $\neg A$-world y for which $x \sim_\Psi y$ (where $\sim_\Psi$ is the symmetric closure of $\preccurlyeq_\Psi$), so that $x \preccurlyeq_{\Psi * A} y$ iff:

    (i) $x \in \min(\preccurlyeq_\Psi, [\![A]\!])$, or

    (ii) $x, y \notin \min(\preccurlyeq_\Psi, [\![A]\!])$ and either

        (a) $x \prec_\Psi y$ or

        (b) $x \sim_\Psi y$ and $(x \in [\![A]\!]$ or $y \in [\![\neg A]\!])$.

---

[4]Note the use of the plural here: we speak of restrained/lexicographic operat*ors*. It is of course customary, in the literature, to refer to *the* restrained/lexicographic operat*or*. However, this way of speaking is only appropriate to the extent that belief states are identifiable with TPO's.

Lexicographic revision operators make every A-world lower ranked than every $\neg A$-world, while preserving the ordering within each of $[\![A]\!]$ and $[\![\neg A]\!]$, so that $x \preccurlyeq_{\Psi * A} y$ iff:

(i) $x \in [\![A]\!]$ and $y \in [\![\neg A]\!]$, or

(ii) $(x \in [\![A]\!]$ iff $y \in [\![A]\!])$ and $x \preccurlyeq_\Psi y$.

Natural revision operators, however, fail to satisfy (P^*) and are thus not members of the family of admissible revision operators. These operators simply promote the minimal A-worlds to be $\preccurlyeq_{\Psi * A}$-minimal, while leaving everything else unchanged, so that $x \preccurlyeq_{\Psi * A} y$ iff:

(i) $x \in \min(\preccurlyeq_\Psi, [\![A]\!])$, or

(ii) $x, y \notin \min(\preccurlyeq_\Psi, [\![A]\!])$ and $x \preccurlyeq_\Psi y$.

Importantly, these three operators share the rather strong property of effectively equating belief states with TPO's. Booth & Chandler [10], however, offered a number of counterexamples to this identification, one of these being the following:

Example 1. *Bashiir and Ayaan have been invited to a party. Initially unsure as to whether either wanted to attend, I now hear that the venue is located too far out of town for either of them. I also believe that they don't get on and are unlikely to attend the same party.*

Example 2. *As above, save that I believe that Bashiir and Ayaan have never met and know nothing about each other.*

Let $A =$ 'Ayaan will attend' and $B =$ 'Bashiir will attend'. With Ψ being the belief state in which I am after having heard of the party's location, Booth & Chandler argue that the following holds: (1) $\neg A \in [\Psi]$ and $\neg B \in [\Psi]$, (2) $\neg A \in [\Psi * B]$ and $\neg B \in [\Psi * A]$, and (3) $A, B \notin [\Psi * A \vee B]$. Assuming L has atoms $\{A, B\}$, AGM dictates that Ψ is associated with the *same* single TPO in both examples, which is represented in Figure 1.

AB	$A\overline{B}$	$\overline{A}B$	$\overline{AB}$
○			
	○	○	
			○

Figure 1: Representation of the unique TPO $\preceq_\Psi$ associated with both Example 1 and Example 2, assuming a language with atomic sentences $\{A, B\}$. The relation $\preceq_\Psi$ orders the worlds from bottom to top, with the minimal world on the lowest level. The columns group worlds according to the sentences that they validate. We write 'AB' for $A \wedge B$' and '$\overline{A}$' for '$\neg A$'.

As Booth & Chandler note, (1)–(3) do *not*, even given AGM and DP, determine whether or not $A \in [(\Psi * A) * B]$. But this underdetermination, they claim, is precisely in order. With respect to Example 1, they argue that, potentially, $\neg A \in [(\Psi * A) * B]$, if the belief Ayaan and Bashiir's mutual dislike is sufficiently deeply entrenched. On the other hand, regarding Example 2, we clearly have $A \in [(\Psi * A) * B]$. If their point is well taken, it follows that the overruling relations of Definition 1, which we have seen to be critical in determining the result of a twofold revision, are not representable in terms of a structure as coarse-grained as a TPO: further information is required.

3. Two principles of non-prioritised revision

The DP postulates, as well as (P*), constrain the relation between a prior conditional belief set on the one hand, and a posterior one on the other. But one might wonder what kinds of constraints govern the relation between *two posterior conditional belief sets* obtained from a common prior by different revisions.

To the best of our knowledge, the only two articles to consider principles of this nature are [12] and, more briefly, [20]. In the former, a slightly more general form of the following pair of syntactic principles is discussed (where the

letter 's' in the name stands for 'strong', to draw a contrast with some weaker versions discussed below):

$(s\beta 1^*)$ If $A \notin [(\Psi * A) * B]$, then $A \notin [(\Psi * C) * B]$

$(s\beta 2^*)$ If $\neg A \in [(\Psi * A) * B]$, then $\neg A \in [(\Psi * C) * B]$

whose semantic counterparts are given by:

$(s\beta 1^*_{\preccurlyeq})$ If $x \in [\![A]\!]$, $y \in [\![\neg A]\!]$ and $y \preccurlyeq_{\Psi * A} x$, then $y \preccurlyeq_{\Psi * C} x$

$(s\beta 2^*_{\preccurlyeq})$ If $x \in [\![A]\!]$, $y \in [\![\neg A]\!]$ and $y \prec_{\Psi * A} x$, then $y \prec_{\Psi * C} x$

On the relative plausibility interpretation of $\preccurlyeq_{\Psi}$, the latter can be informally glossed as follows: if (i) there exists some potential evidence, consistent with a world x but not with a world y, such that x would be considered no more plausible than (respectively: strictly less plausible than) y after receiving it, then (ii) there is no potential evidence *whatsoever* that would lead x to be considered more plausible than (respectively: at least as plausible as) y.

It is easy to see that, on the assumption that $\preccurlyeq_{\Psi * \top} = \preccurlyeq_{\Psi}$ (which follows from $(C1^*_{\preccurlyeq})$), these respectively generalise $(C3^*_{\preccurlyeq})$ and $(C4^*_{\preccurlyeq})$, which correspond to the special cases in which C is a tautology.

These postulates can be interpreted in terms of the overruling and strict overruling introduced above, in Definition 1. $(s\beta 1^*)$ tells us that, if B overrules A in Ψ, then A will not be believed following any sequence of two revisions starting in Ψ ending with B, while $(s\beta 2^*)$ says that, if B strictly overrules A in Ψ, then A will be rejected following any such sequence of two revisions.

4. Proper ordinal intervals for non-prioritised revision

We remarked above that it was a more *general* form of $(s\beta 1^*)$, $(s\beta 2^*)$ and their semantic counterparts that interested Booth & Meyer. The reason for this is that their topic of interest was not in fact $*$, but rather a more *general* kind of operator: a *non-prioritised* 'revision' operator $\circ$, which does not necessarily

satisfy the Success postulate. They showed that these operators could be represented as relating belief states associated with a particular type of structure.

The general idea behind their construction is that two-step revision ought to be guided, not by an ordering of the mere set of *worlds* W, but by a particular kind of ordering of a corresponding set of *signed worlds* $W^\pm = \{w^i \mid w \in W \text{ and } i \in \{-, +\}\}$. Depending on the input to the first revision, each world w finds itself represented in this ordering by either a 'positive' counterpart w^+ (in the event that it validates the input) or a 'negative' counterpart w^- (in the event that it invalidates it). The posterior ordering of worlds, which determines the outcome of the second revision, is then given by the ordering of the relevant signed worlds.

More formally, let us offer the following key definitions:

Definition 2. $\leq$ *is a* proper ordinal interval (POI) assignment *to W if it is a relation over the corresponding set of signed worlds $W^\pm := \{w^i \mid w \in W \text{ and } i \in \{-, +\}\}$ such that:*

(≤ 1) $\quad \leq$ *is a TPO*

(≤ 2) $\quad x^+ < x^-$

(≤ 3) $\quad x^+ \leq y^+$ *iff* $x^- \leq y^-$.[5]

---

[5]Peppas & Williams [21] have suggested a connection here to the notion of a *semiorder*. This concept has a long history of applications in the cognitive and decision sciences (see [22] for a book-length treatment) and has recently surfaced in the belief revision literature, in the context of the semantic representation of certain weakenings of the AGM postulates ([23, 24, 21]). The class of semiorders subsumes that of TPO's, relaxing the condition of transitivity of indifference. The connection to the present framework is presumably the following: as is well known, semiorders are representable by mapping elements onto the members of a set of constant-length intervals on the real line, such that $x \prec y$ if the interval associated with x lies strictly to the left of the one associated with y and $x \sim y$ otherwise. POI assignments correspond to classes of such mappings, closed under order-preserving transformations. So every POI assignment generates a unique semiorder. However, the converse, of course, does not hold. Indeed, the representation of semiorders is unique up to a range of transformations that is broader than the order-preserving ones. In particular, any two interval representations

Definition 3. *Where $\preccurlyeq$ is a TPO over W and $\leq$ is a POI assignment to W, we say that $\leq$ is* faithful *to $\preccurlyeq$ if it satisfies:*

(≤ 4) $\quad x^+ \leq y^+$ *iff* $x \preccurlyeq y$.

We write (i) $x^\delta < y^\epsilon$ if $x^\delta \leq y^\epsilon$ but $y^\epsilon \not\leq x^\delta$ and (ii) $x^\delta \dot{=} y^\epsilon$ if $x^\delta \leq y^\epsilon$ and $y^\epsilon \leq x^\delta$.

Booth & Meyer assumed that each belief state Ψ is associated, not only with a TPO $\preccurlyeq_\Psi$, but with a POI assignment $\leq_\Psi$ that is faithful to $\preccurlyeq_\Psi$ (they remained agnostic as to whether states are to be identified with POI assignments; we will follow suit). This assignment was then taken to determine the agent's posterior TPO upon revision by A, i.e. $\preccurlyeq_{\Psi \circ A}$, in the following manner:

Definition 4. $\circ$ *is a* non-prioritised POI revision operator *if $\circ$ is a function from state-sentence pairs to states, such that for every state Ψ there is a POI assignment $\leq_\Psi$ such that, for any sentence A, $x \preccurlyeq_{\Psi \circ A} y$ iff $r_A(x) \leq_\Psi r_A(y)$, where*

$$
r_A(x) = \begin{cases} x^+ & \text{if } x \in [\![A]\!] \\ \\ x^- & \text{if } x \in [\![\neg A]\!]. \end{cases}
$$

We can see from this that, as we stated above, the informal interpretation of the semantics is that the positive and negative counterparts of a world represent its position in the light of auspicious and inauspicious inputs, respectively. For a given revision input A and world x, if $x \in [\![A]\!]$ the 'news is good' for x. If $x \in [\![\neg A]\!]$, the 'news is bad'.

---

in which the intervals associated with x and y overlap will map onto semiorders for which $x \sim y$. So there is no distinction drawn, for instance, between (i) a representation of the relation between x and y in which these elements are mapped onto identical intervals and (ii) a representation in which they are mapped onto distinct but overlapping intervals. But this is a distinction that typically has repercussions for iterated revision in Booth & Meyer's scheme, as well as the one that we offer later on in the paper.

General forms of our principles $(s\beta1^*_{\preccurlyeq})$ and $(s\beta2^*_{\preccurlyeq})$ turn out to play a key role in this model. Indeed, Booth & Meyer [12, Theorem 1] show that $\circ$ is a non-prioritised POI revision operator if and only if it satisfies $(C1^\circ_{\preccurlyeq})$, $(C2^\circ_{\preccurlyeq})$, $(P^\circ_{\preccurlyeq})$, $(s\beta1^\circ_{\preccurlyeq})$ and $(s\beta2^\circ_{\preccurlyeq})$, where these principles are obtained from their counterparts for (prioritised) revision in the obvious manner, by substituting the $\circ$ symbol for $*$.

Non-prioritised POI revision operators can helpfully be understood diagrammatically. Figure 2 represents two proper ordinal interval assignments that are both faithful to the same TPO, as can be seen by the ordering of the endpoints. Figure 3 represents, by means of the filled circles, the TPO resulting from the non-prioritised revisions of the corresponding states by A. The left hand diagram in that figure also illustrates failure of Success, since two of the three minimal worlds in the posterior TPO are in $[\![\neg A]\!]$: sometimes the 'good news' for the worlds that validate the input is simply not 'good enough'.

We note that lexicographic revision operators are special cases of this family in which $x^+ <_\Psi y^-$ for all $x, y \in W$.[6]

5. The principles in a prioritised setting

In spite of their arguable appropriateness in a non-prioritised setting, $(s\beta1^*_{\preccurlyeq})$ and $(s\beta2^*_{\preccurlyeq})$ prove to be problematically strong when one imposes Success. For one, it turns out that, in such a context the only kind of operators satisfying $(s\beta1^*_{\preccurlyeq})$ are lexicographic revision operators, and hence that $(s\beta1^*_{\preccurlyeq})$ imposes the reductionist assumption that we have suggested is objectionable. Indeed:

Theorem 1. *Let $*$ be a revision operator satisfying AGM and $(s\beta1^*)$. Then it also satisfies the Recalcitrance property [6]:*

(Rec^*) *If $A \wedge B$ is consistent, then $A \in [(\Psi * A) * B]$.*

---

[6]As an anonymous referee for this journal has reminded us, there are also connections between Booth & Meyer's proposal and the class of 'improvement' operators' of [25] and [26]. These connections are discussed in some detail in [12, Section 9].
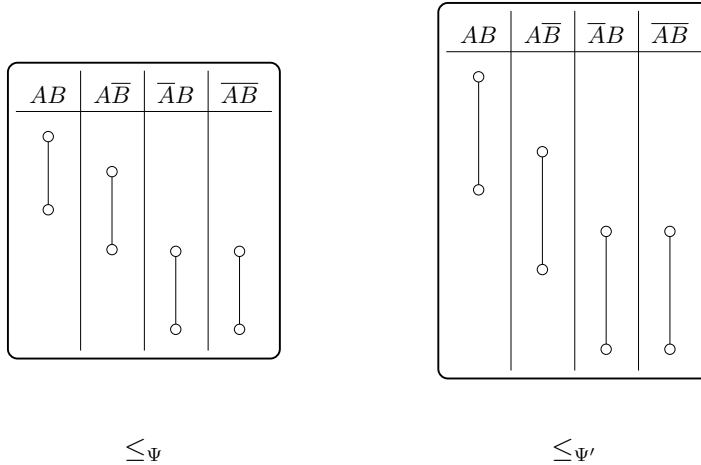
Figure 2: Distinct POI assignments, respectively associated with states Ψ and Ψ', that are faithful to a same TPO. The bottom and top interval endpoints respectively represent the positive and negative counterparts x^+ and x^- of each world x.

Proof: If $A \wedge B$ is consistent, then $A \in [(\Psi * A \wedge B) * B]$ from AGM. Then $A \in [(\Psi * A) * B]$ by $(s\beta1^*)$.[7]   □

Since we know (see, e.g., [12, 6]), that lexicographic revision operators are the only admissible operators satisfying (Rec^*), we obtain the following corollary, which also gives us an alternative characterisation of lexicographic revision operators:

Corollary 1. *The only operators satisfying AGM,* (C1^*), (C2^*) *and* $(s\beta1^*)$ *are lexicographic revision operators.*[8]

---

[7]If one assumes consistency of revision inputs, it is trivial to show that the implication also runs the other way, so that (Rec^*) and $(s\beta1^*)$ are then equivalent, given AGM: Suppose $A \in [(\Psi * C) * B]$. Since we thereby implicitly assume consistency of B, $A \wedge B$ must also be consistent (as, by AGM, $A \wedge B \in [(\Psi * C) * B]$ and $[(\Psi * C) * B]$ is consistent). Hence, by (Rec^*), $A \in [(\Psi * A) * B]$.

[8]What about $(s\beta2^*_{\preccurlyeq})$? Lexicographic revision satisfies it trivially, since it satisfies: If

$$\preccurlyeq_{\Psi \circ A} \qquad\qquad \preccurlyeq_{\Psi' \circ A}$$
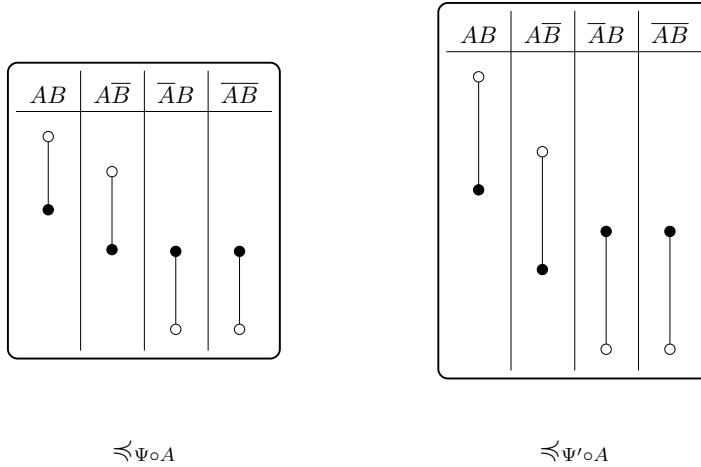
Figure 3: Posterior TPO's resulting from non-prioritised revision by A of the states Ψ and Ψ' in Figure 2.

These principles also face a class of direct counterexamples that match the following general pattern: (i) A provides a defeasible reason to believe $\neg B$ and (ii) C is equivalent to the conjunction of A and a defeater for A's support for $\neg B$. Under these conditions, it can plausibly be the case that $\neg A \in [(\Psi * A) * B]$ but $A \in [(\Psi * C) * B]$, contradicting both principles. What follows is an example of a trio of sentences that fit the bill.

Example 3. *Let A = 'She is a pro archer', B = 'She missed the target at 5 yards' and C = 'She is a pro archer but isn't wearing her glasses'.*

This negative result raises the following question: Is there any way to weaken $(s\beta1^*_{\preccurlyeq})$ and $(s\beta2^*_{\preccurlyeq})$ to allow a wider, but intuitively plausible, family of iterated prioritised revision operators? The answer, as we will now show, is 'yes'.

$x \in [\![A]\!]$, $y \in [\![\neg A]\!]$, then $x \prec_{\Psi * A} y$. We can analogously show that it implies, given AGM, the following weakening of (Rec*): If $A \wedge B$ is consistent, then $\neg A \notin [(\Psi * A) * B]$. But this is too weak to allow us to recover (Rec*) and indeed, $(s\beta2^*_{\preccurlyeq})$ is not uniquely satisfied by lexicographic revision.

6. Success via naturalisation

Our guiding idea is to take the family of operators discussed in Section 4 and ensure satisfaction of Success, not by adding the principle to the list of characteristic postulates (we have just seen that this is a non-starter) but rather by *minimally transforming* the TPO associated with the posterior belief state in such a way that the input to revision finds itself included in the posterior belief set. More specifically, we are seeking a TPO transformation that minimises, subject to the constraint of Success, the symmetric difference distance d_S, given by:

Definition 5. $d_S(\preccurlyeq, \preccurlyeq') := |(\preccurlyeq - \preccurlyeq') \cup (\preccurlyeq' - \preccurlyeq)|.$

so that $d_S(\preccurlyeq, \preccurlyeq')$ counts the disagreements over relations of weak preference between the two orderings, returning the number of pairs that are in $\preccurlyeq$ but not in $\preccurlyeq'$ and vice versa.

To state our first significant result, let us introduce some notation and terminology:

Definition 6. *For all functions $*$ and $\circ$ from state-sentence pairs to states, $*$ is a* naturalisation *of $\circ$ if for all states Ψ and sentences A in L:*

$x \preccurlyeq_{\Psi * A} y$ *iff either*

     (i) $x \in \min(\preccurlyeq_{\Psi \circ A}, [\![A]\!])$, or

     (ii) $x, y \notin \min(\preccurlyeq_{\Psi \circ A}, [\![A]\!])$ and $x \preccurlyeq_{\Psi \circ A} y$.

$\mathbb{N}(*, \circ)$ *means that the couple $(*, \circ)$ satisfies the relation defined above.*

Recalling the definition of natural revision in Section 2.3, we can equivalently say that $*$ is a naturalisation of $\circ$ iff it is the *composition* of a non-prioritised POI revision operator $\circ$ and a natural revision operator $\boxplus$, so that:

$$\preccurlyeq_{\Psi * A} = \preccurlyeq_{(\Psi \circ A) \boxplus A}$$

It can be shown that the transformation, by means of a naturalisation step, of the TPO associated with the posterior belief state $\Psi \circ A$ is 'minimal' in the required sense:

Proposition 1. $\mathbb{N}(*, \circ)$ *iff, for all $A \in L$, $\preccurlyeq_{\Psi * A}$ minimises the symmetric difference distance d_S to $\preccurlyeq_{\Psi \circ A}$, subject to the constraints of AGM.*

In view of this, we now offer the following definition:

Definition 7. $*$ *is a* proper interval order (POI) *revision operator if $\mathbb{N}(*, \circ)$ for some non-prioritised POI revision operator $\circ$.*

and propose that rational revision is POI revision.

Our suggestion generalises one that was made in [7], in which restrained revision operators were shown to be naturalisations of a particular class of non-prioritised revision operators due to Papini [27]. Indeed, the latter satisfy: $x \preccurlyeq_{\Psi \circ A} y$ iff (a) $x \prec_\Psi y$ or (b) $x \sim_\Psi y$ and ($x \in [\![A]\!]$ or $y \in [\![\neg A]\!]$). These conditions, of course, simply correspond to (ii)(a) and (ii)(b) in the definition of restrained revision operators given in Section 2. The proposal is also somewhat reminiscent of the manner in which the Levi Identity [28] treats non-iterated revision as the composition of a contraction and an expansion ($[\Psi * A] = [\Psi \div \neg A] + A$, where $[\Psi \div \neg A] + A := \mathrm{Cn}([\Psi \div \neg A] \cup \{A\})$), with our natural revision step $\boxplus$ playing the role of the expansion step $+$.

Figure 4 provides a general overview of the model, with the various arrows denoting functional determination. From bottom to top, each belief state Ψ is mapped onto a POI assignment $\leq_\Psi$. This POI assignment determines a TPO $\preccurlyeq_\Psi$, such that $x \preccurlyeq_\Psi y$ iff $x^+ \leq_\Psi y^+$. Finally the TPO in turn determines a belief set $[\Psi]$, such that $A \in [\Psi]$ iff $\min(\preccurlyeq_\Psi, W) \subseteq [\![A]\!]$. These mappings are potentially many-to-one, so that we obtain increasingly coarse descriptions of an agent's beliefs as one moves upwards. From left to right, the function $\circ$ maps the prior belief state Ψ onto an 'intermediate' state $\Psi \circ A$, before the function $\boxplus$ maps the latter onto the posterior state $\Psi * A = (\Psi \circ A) \boxplus A$.
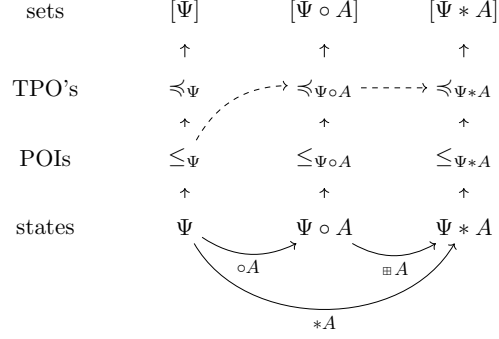
Figure 4: Functional dependencies in POI revision

We have used dashed arrows to denote some further functional dependencies. The constraints of [12] ensure that the prior POI assignment $\leq_\Psi$ determines the 'intermediate' TPO $\preccurlyeq_{\Psi \circ A}$. Finally, the constraints operating on the function $\boxplus$ ensure that *this* in turn determines the posterior TPO $\preccurlyeq_{\Psi * A}$. This last step is achieved by moving the $\preccurlyeq_{\Psi \circ A}$-minimal A-worlds to the bottom of the ordering. See Figure 5 for illustration.

The naturalisation step ensures that we have $B \in [\Psi * A]$ iff $\min(\preccurlyeq_\Psi, [\![A]\!]) \subseteq [\![B]\!]$, so AGM will now clearly be satisfied, including Success. Furthermore, the following general fact about naturalisation establishes that the set of POI revision operators is a subset of the set of admissible operators:

Proposition 2. *For any functions $\circ$ and $*$ from state-sentence pairs to states, such that $\mathbb{N}(*, \circ)$, if $\circ$ satisfies $(\text{Eq}^\circ_{\preccurlyeq})$, $(\text{C1}^\circ_{\preccurlyeq})$, $(\text{C2}^\circ_{\preccurlyeq})$ and $(\text{P}^\circ_{\preccurlyeq})$, then $*$ will satisfy $(\text{Eq}^*_{\preccurlyeq})$, $(\text{C1}^*_{\preccurlyeq})$, $(\text{C2}^*_{\preccurlyeq})$ and $(\text{P}^*_{\preccurlyeq})$.*

Indeed, we have already noted that non-prioritised POI revision operators satisfy $(\text{C1}^\circ_{\preccurlyeq})$, $(\text{C2}^\circ_{\preccurlyeq})$ and $(\text{P}^\circ_{\preccurlyeq})$. Furthermore, Booth & Meyer show that they also satisfy $(\text{Eq}^\circ_{\preccurlyeq})$.

The family of POI revision operators includes some familiar figures:
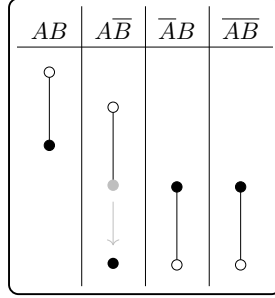
$$\leq_{\Psi * A}$$

Figure 5: Posterior TPO $\leq_{\Psi * A}$ obtained via naturalisation of the TPO $\preccurlyeq_{\Psi \circ A}$ from Figure 3.

Proposition 3. *Both lexicographic and restrained revision operators are POI revision operators.*

Indeed, we have pointed out, at the end of Section 4, that lexicographic revision operators are themselves non-prioritised POI revision operators. Furthermore, since they satisfy Success, they will be identical with their own naturalisations. Regarding restrained revision operators, the result was established in Proposition 14 of [7]: they are, as we noted above, naturalisations of Papini's 'reverse' lexicographic revision operators, which are non-prioritised POI revision operators.
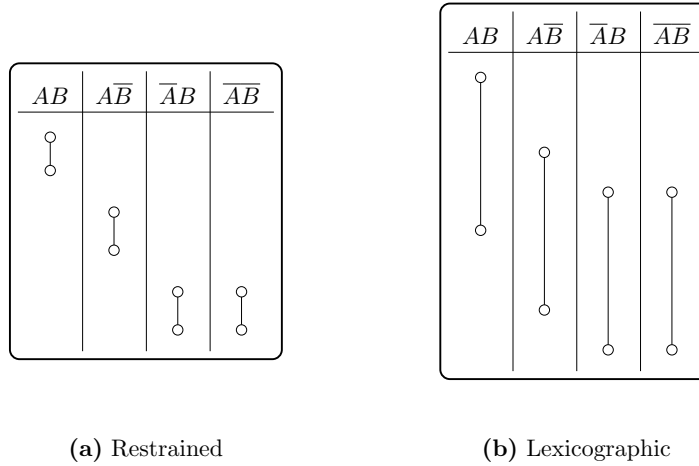
Lexicographic revision can be represented by a restriction on the set of permissible POI assignments, such that the following is satisfied:

(a) $\forall x, y \in W$, $x \prec_\Psi y$ iff $x^+ <_\Psi y^+ <_\Psi x^-$

(b) $\forall x, y \in W$, $x \sim_\Psi y$ iff $x^+ \doteq_\Psi y^+$

To obtain restrained revision, (a) is to be replaced by:

(a') $\forall x, y \in W$, $x \prec_\Psi y$ iff $x^- <_\Psi y^+$

20

See Figure 6 for a visual representation.[9, 10]

(a) Restrained   (b) Lexicographic

Figure 6: Representations of restrained and lexicographic revision operators in terms of alternative POI assignments.

We close this section by noting that the representation of two-step revision dispositions in terms of POI assignments is unique only up to a certain type of (admittedly reasonably modest) transformation. To express our result, it is convenient to introduce the following terminology:

Definition 8. $\leq$ *and* $\leq'$ agree *on a set* S *if* $\leq \cap (S \times S) =\leq' \cap (S \times S)$

With this in hand, we can offer:

---

[9]Note that while the principles of POI revision generally only constrain two steps of revision, since they only provide a recipe for obtaining a posterior *TPO* from a prior POI assignment, in these two special cases, the posterior TPO itself determines the posterior *POI assignment*, via (a) and (b) or (a') and (b).

[10]Besides lexicographic and restrained revision operators, which identify states with TPO's, there are of course many other operators in the POI family that allow for a 'concrete' representation of states. In Appendix A, we provide an example, in which the state is this time represented by a mapping of worlds onto the real numbers. We are grateful to an anonymous reviewer for suggesting to include such an example in the paper.

Proposition 4. *Let Ψ and Ψ' be two belief states and $*$ a POI revision operator. Then the following two statements are equivalent:*
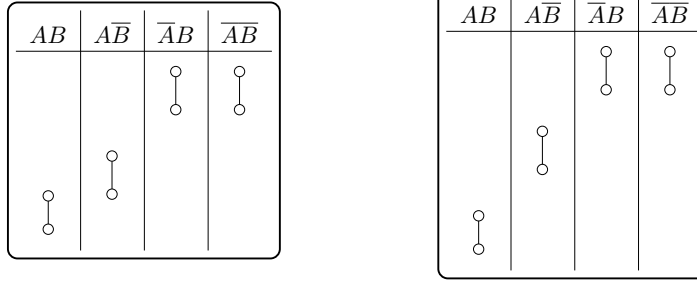
(1) $\forall A \in L, \preccurlyeq_{\Psi * A} = \preccurlyeq_{\Psi' * A}$

(2) (a) $\preccurlyeq_{\Psi'} = \preccurlyeq_{\Psi} = \preccurlyeq$, and (b) $\leq_{\Psi'}$ and $\leq_{\Psi}$ agree on $\{x^-, y^+\}$ for all $x, y \in W$ such that

(i) $x \prec y$ and

(ii) there exists $z \in W$ such that $z \neq x$ and $z \prec y$.

Figure 7 provides an example of two distinct POI assignments yielding the same posterior TPO after revision by a certain sentence (and hence related by the relevant transformation).

Figure 7: Two distinct POI assignments yielding the same posterior TPO for all inputs to revision.

7. Two weaker principles

It is easy to see that neither $(s\beta 1^*_{\preccurlyeq})$ nor $(s\beta 2^*_{\preccurlyeq})$ are generally satisfied by POI revision operators. Indeed, let $W = \{x, y, z\}$ and $\leq_{\Psi}$ be given as follows: $z^+ <_{\Psi} y^+ <_{\Psi} z^- <_{\Psi} y^- <_{\Psi} x^+ <_{\Psi} x^-$. Then $y \prec_{\Psi * x \vee z} x$, but $x \prec_{\Psi * x} y$. However, as we shall see from Proposition 8 in the next section, we *do* nevertheless obtain the following weakened versions of these principles, which incorporate into their antecedents the further requirement that $x \notin \min(\preccurlyeq_{\Psi}, [\![C]\!])$:

($\beta1^*_\preccurlyeq$)   If $x \notin \min(\preccurlyeq_\Psi, \llbracket C \rrbracket)$, $x \in \llbracket A \rrbracket$, $y \in \llbracket \neg A \rrbracket$, and $y \preccurlyeq_{\Psi * A} x$, then $y \preccurlyeq_{\Psi * C} x$

($\beta2^*_\preccurlyeq$)   If $x \notin \min(\preccurlyeq_\Psi, \llbracket C \rrbracket)$, $x \in \llbracket A \rrbracket$, $y \in \llbracket \neg A \rrbracket$, and $y \prec_{\Psi * A} x$, then $y \prec_{\Psi * C} x$

Regarding the syntactic counterparts of these principles:

Proposition 5. (a) *Given AGM, ($\beta1^*_\preccurlyeq$) is equivalent to:*

($\beta1^*$)   *If $A \notin [(\Psi * A) * B]$ and $B \to \neg A \in [\Psi * C]$, then $A \notin [(\Psi * C) * B]$*

(b) *Given AGM, ($\beta2^*_\preccurlyeq$) is equivalent to:*

($\beta2^*$)   *If $\neg A \in [(\Psi * A) * B]$ and $B \to \neg A \in [\Psi * C]$, then $\neg A \in [(\Psi * C) * B]$*

These principles are particularly interesting insofar as they avoid the kind of counterexample to (s$\beta1^*_\preccurlyeq$) and (s$\beta2^*_\preccurlyeq$) that we raised earlier (see Example 1 and Example 2). Recall that we considered the three following sentences: $A = $ 'She is a pro archer', $B = $ 'She missed the target at 5 yards' and $C = $ 'She is a pro archer but isn't wearing her glasses'. We noted that, plausibly, $\neg A \in [(\Psi * A) * B]$, while $A \in [(\Psi * C) * B]$, contradicting our stronger pair of principles. However, note that, intuitively, $B \to \neg A \notin [\Psi * C]$: after finding out that she is a pro archer who isn't wearing her glasses, we ought not believe that if she missed the target, then she isn't a pro archer. Given this, neither of the weaker ($\beta1^*$) and ($\beta2^*$) can be applied to yield the problematic consequence that, if $\neg A \in [(\Psi * A) * B]$, then $A \notin [(\Psi * C) * B]$.

It will turn out to be useful, in the final sections of the paper, to have noted the following equivalent formulations of ($\beta1^*_\preccurlyeq$) and ($\beta2^*_\preccurlyeq$):

Proposition 6. (a) *Given (C2$^*_\preccurlyeq$) and (C4$^*_\preccurlyeq$), ($\beta1^*_\preccurlyeq$) is equivalent to the conjunction of the following two principles:*

($\gamma1^*_\preccurlyeq$)   *If $x \in \llbracket A \rrbracket$, $y \in \llbracket \neg A \rrbracket$ and $y \preccurlyeq_{\Psi * A} x$, then $y \preccurlyeq_{\Psi * A \vee C} x$*

($\gamma3^*_\preccurlyeq$)   *If $x \notin \min(\preccurlyeq_\Psi, \llbracket C \rrbracket)$, $x \in \llbracket A \vee C \rrbracket$, $y \in \llbracket \neg(A \vee C) \rrbracket$, and $y \preccurlyeq_{\Psi * A \vee C} x$, then $y \preccurlyeq_{\Psi * C} x$.*

(b) *Given* $(C1^*_\preccurlyeq)$ *and* $(C3^*_\preccurlyeq)$, $(\beta2^*_\preccurlyeq)$ *is equivalent to the conjunction of the following two principles:*

$(\gamma2^*_\preccurlyeq)$ *If* $x \in [\![A]\!]$, $y \in [\![\neg A]\!]$ *and* $y \prec_{\Psi*A} x$, *then* $y \prec_{\Psi*A\vee C} x$

$(\gamma4^*_\preccurlyeq)$ *If* $x \notin \min(\preccurlyeq_\Psi, [\![C]\!])$, $x \in [\![A \vee C]\!]$, $y \in [\![\neg(A \vee C)]\!]$ *and* $y \prec_{\Psi*A\vee C} x$,
then $y \prec_{\Psi*C} x$.

Note that, given the assumption that $\preccurlyeq_{\Psi*\top} = \preccurlyeq_\Psi$, which follows from $(C1^*_\preccurlyeq)$, $(\gamma1^*_\preccurlyeq)$ and $(\gamma2^*_\preccurlyeq)$ respectively entail $(C3^*_\preccurlyeq)$ and $(C4^*_\preccurlyeq)$ (let $C = \neg A$). However, none of these four new principles, and hence neither of $(\beta1^*_\preccurlyeq)$ and $(\beta2^*_\preccurlyeq)$, are generally sound for admissible operators:

Proposition 7. *None of* $(\gamma1^*_\preccurlyeq)$ *to* $(\gamma4^*_\preccurlyeq)$ *follows from AGM,* $(C1^*_\preccurlyeq)$, $(C2^*_\preccurlyeq)$ *and* $(P^*_\preccurlyeq)$ *alone.*

8. Characterisations of POI operators

We have now identified a number of sound principles for the class of POI revision operators, which, we would like to remind the reader, subsumes both restrained and lexicographic operators. Next, we would like to *characterise* it.

8.1. Semantic characterisation

For our semantic characterisation, we need to introduce three more postulates, the first two of which are respective strengthenings of $(\beta1^*_\preccurlyeq)$ and $(\beta2^*_\preccurlyeq)$, which can be recovered by setting $z = y$:

$(\alpha1^*_\preccurlyeq)$ If $x \notin \min(\preccurlyeq_\Psi, [\![C]\!])$, $x \in [\![A]\!]$, $y \in [\![\neg A]\!]$, $z \preccurlyeq_\Psi y$ and $y \preccurlyeq_{\Psi*A} x$, then
$z \preccurlyeq_{\Psi*C} x$

$(\alpha2^*_\preccurlyeq)$ If $x \notin \min(\preccurlyeq_\Psi, [\![C]\!])$, $x \in [\![A]\!]$, $y \in [\![\neg A]\!]$, $z \preccurlyeq_\Psi y$ and $y \prec_{\Psi*A} x$, then
$z \prec_{\Psi*C} x$

$(\alpha3^*_\preccurlyeq)$ If $x \notin \min(\preccurlyeq_\Psi, [\![C]\!])$, $x \in [\![A]\!]$, $y \in [\![\neg A]\!]$, $z \prec_\Psi y$ and $y \preccurlyeq_{\Psi*A} x$, then
$z \prec_{\Psi*C} x$

It can be shown that

Proposition 8. $(\alpha 1^*_{\preccurlyeq})$, $(\alpha 2^*_{\preccurlyeq})$ and $(\alpha 3^*_{\preccurlyeq})$ are satisfied by POI revision operators.

These principles can perhaps be viewed as qualified pseudo-'transitivity' principles, if one ignores the subscripts. We note, furthermore, that:

Proposition 9. $(\alpha 1^*_{\preccurlyeq})$ and $(\alpha 2^*_{\preccurlyeq})$ are equivalent to the conjunctions of $(\beta 1^*_{\preccurlyeq})$ and $(\beta 2^*_{\preccurlyeq})$, respectively, with the following semantic principles, again respectively:

$(\beta 3^*_{\preccurlyeq})$ If $z \neq y$, $x \notin \min(\preccurlyeq_\Psi, \llbracket C \rrbracket)$, $x \in \llbracket A \rrbracket$, $y \in \llbracket \neg A \rrbracket$, $z \preccurlyeq_\Psi y$, and $y \preccurlyeq_{\Psi * A} x$, then $z \preccurlyeq_{\Psi * C} x$

$(\beta 4^*_{\preccurlyeq})$ If $z \neq y$, $x \notin \min(\preccurlyeq_\Psi, \llbracket C \rrbracket)$, $x \in \llbracket A \rrbracket$, $y \in \llbracket \neg A \rrbracket$, $z \preccurlyeq_\Psi y$, and $y \prec_{\Psi * A} x$, then $z \prec_{\Psi * C} x$
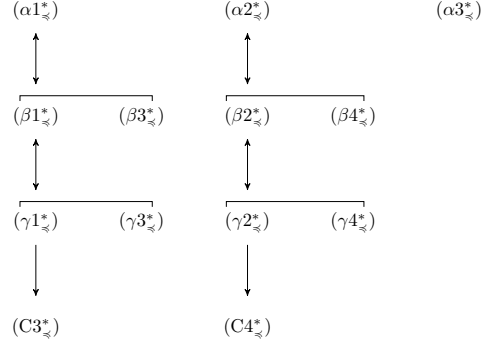
So Proposition 8 shows that $(\beta 1^*_{\preccurlyeq})$ and $(\beta 2^*_{\preccurlyeq})$–and hence, in view of Proposition 6, $(\gamma 1^*_{\preccurlyeq})$ to $(\gamma 4^*_{\preccurlyeq})$–are sound for POI revision operators. We can now present our main result, which is a semantic characterisation of the family:

Theorem 2. $*$ is a POI revision operator iff it satisfies AGM, $(Eq^*_{\preccurlyeq})$, $(C1^*_{\preccurlyeq})$, $(C2^*_{\preccurlyeq})$, $(P^*_{\preccurlyeq})$, $(\alpha 1^*_{\preccurlyeq})$, $(\alpha 2^*_{\preccurlyeq})$, and $(\alpha 3^*_{\preccurlyeq})$.

In conjunction with the results of Booth & Meyer regarding non-prioritised POI revision operators, Propositions 2 and 8 establish the left-to-right direction of the above claim. For the other direction we need to show that, if $*$ satisfies the relevant semantic properties, then there exists a non-prioritised POI revision operator $\circ$ such that $\mathbb{N}(*, \circ)$. The construction works as follows: From $*$, define $\circ$ by setting, for all $x, y \in W$, $x \preccurlyeq_{\Psi \circ A} y$ iff $x \preccurlyeq_{\Psi * A \vee \neg(x \vee y)} y$. Given $(Eq^*_{\preccurlyeq})$, $(C1^*_{\preccurlyeq})$ and $(C2^*_{\preccurlyeq})$ this is equivalent to:

$$x \preccurlyeq_{\Psi \circ A} y \text{ iff } \begin{cases} x \preccurlyeq_\Psi y & \text{if } x \sim^A y \\[2mm] x \preccurlyeq_{\Psi * \neg y} y & \text{if } x \vartriangleleft^A y \\[2mm] x \preccurlyeq_{\Psi * \neg x} y & \text{if } y \vartriangleleft^A x \end{cases}$$

$(\alpha1^*_{\preccurlyeq})$ $(\alpha2^*_{\preccurlyeq})$ $(\alpha3^*_{\preccurlyeq})$

$(\beta1^*_{\preccurlyeq})$ $(\beta3^*_{\preccurlyeq})$ $(\beta2^*_{\preccurlyeq})$ $(\beta4^*_{\preccurlyeq})$

$(\gamma1^*_{\preccurlyeq})$ $(\gamma3^*_{\preccurlyeq})$ $(\gamma2^*_{\preccurlyeq})$ $(\gamma4^*_{\preccurlyeq})$

$(C3^*_{\preccurlyeq})$ $(C4^*_{\preccurlyeq})$

Figure 8: Logical relations between some of the semantic postulates discussed above. Note that certain implications hinge on some of the Darwiche & Pearl postulates.

where (i) $x \trianglelefteq^A y$ iff $x \in [\![A]\!]$ or $y \in [\![\neg A]\!]$, (ii) $x \sim^A y$ iff $x \trianglelefteq^A y$ and $y \trianglelefteq^A x$, (iii) $x \triangleleft^A y$ iff $x \trianglelefteq^A y$ but not $y \trianglelefteq^A x$.[11]

8.2. Two syntactic characterisations

In this part we offer two different syntactic characterisations of the family of POI revision operators. The first involves the following postulates:

($\Omega1^*$)   If $\neg A \notin [\Psi * A \vee B]$ and $A \notin [(\Psi * A) * B]$, then $B \notin [(\Psi * B) * A]$

($\Omega2^*$)   If $\neg A \notin [\Psi * A \vee B]$ and $\neg A \in [(\Psi * A) * B]$, then $\neg B \in [(\Psi * B) * A]$

---

[11]Note that, while we explicitly claim that states ought to be associated with something more fine-grained than TPO's, namely POI assignments, the semantic principles that characterize POI revision are framed in terms of relations between TPO's. An anonymous reviewer for this journal has suggested that some readers could find this puzzling.

But there is no tension here, much in the same way that there is no tension between characterizing AGM-compliant revision operators in terms of principles that constrain relations between belief sets, while explicitly claiming that states ought to be identified with something more fine-grained than belief sets (e.g. TPO's, POI assignments, etc.).

$(\Omega 3^*)$ If $\neg B \in [\Psi * A \vee B]$ and $A \notin [(\Psi * A) * B]$, then $\neg B \in [(\Psi * B) * A]$

These principles admit an interpretation in terms of the notions of overruling and strict overruling that we introduced in Definition 1. Indeed, $(\Omega 1^*)$ and $(\Omega 2^*)$ stipulate conditions under which the obtaining of these relations entail that of their converses, while $(\Omega 3^*)$ offers a condition that is sufficient for B's overruling A to entail A's *strictly* overruling B. More specifically, $(\Omega 1^*)$ tells us that, if A is at least as well entrenched in the agent's beliefs as B[12] and B overrules A, then A overrules B. $(\Omega 2^*)$ tells us that, under the same entrenchment condition, the analogous implication holds for strict overruling. Finally, $(\Omega 3^*)$ translates as the claim that, if A is strictly better entrenched than B in the agent's beliefs[13] and B overrules A, then A strictly overrules B.

Our first syntactic characterisation is then given by the following:

Proposition 10. $*$ *is a POI revision operator iff it satisfies AGM,* (Eq^*), (C1^*), (C2^*), $(\beta 1^*)$, $(\beta 2^*)$, $(\Omega 1^*)$–$(\Omega 3^*)$.

While it employs some fairly accessible principles, this result 'bundles' the contribution of $(\text{P}^*_{\preccurlyeq})$ into the principles $(\Omega 1^*)$–$(\Omega 3^*)$. For this reason, we offer a second characterisation that separates out the contributions and maps each characteristic semantic principle onto a corresponding syntactic counterpart.

It turns out that the exact syntactic counterparts of $(\beta 3^*_{\preccurlyeq})$, $(\beta 4^*_{\preccurlyeq})$, and $(\alpha 3^*_{\preccurlyeq})$ are given as follows, where $\veebar$ denotes exclusive disjunction (XOR):

Proposition 11. (a) *Given AGM,* $(\beta 3^*_{\preccurlyeq})$ *is equivalent to*

---

[12] The notion of comparative entrenchment is perhaps best understood in terms of the operation of contraction $\div$: A is at least as well entrenched in the agent's beliefs as B iff the agent would at least give up B were he or she have to give up at least one of A or B $(B \notin [\Psi \div A \wedge B])$. This is equivalent, given the Levi and Harper Identities, to the following definition in terms of revision: A is at least as well entrenched in the agent's beliefs as B iff the agent would at least give up $\neg A$ were he or she have to come to believe $A \vee B$ $(\neg A \notin [\Psi * A \vee B])$.

[13] A is strictly better entrenched in the agent's beliefs than B iff $A \in [\Psi \div A \wedge B]$. In terms of revision, this last condition translates into $\neg B \in [\Psi * A \vee B]$.

($\beta3^*$)   If $B_2 \notin [\Psi * B_1]$, $B_1 \to A \notin [(\Psi * A) * B_2]$, and $B_2 \to \neg A \in [\Psi * C]$,
     then $B_2 \wedge A \notin [(\Psi * C) * B_1 \veebar B_2]$.

(b) *Given AGM and* $(C4^*_\preccurlyeq)$, $(\beta4^*_\preccurlyeq)$ *is equivalent to:*

($\beta4^*$)   If $B_2 \notin [\Psi * B_1]$, $B_1 \wedge \neg A \in [(\Psi * A) * B_2]$, and $B_2 \to \neg A \in [\Psi * C]$,
     then $B_2 \to \neg A \in [(\Psi * C) * B_1 \veebar B_2]$.

(c) *Given AGM and* $(C3^*_\preccurlyeq)$, $(\alpha3^*_\preccurlyeq)$ *is equivalent to:*

($\alpha3^*$)   If $\neg B_2 \in [\Psi * B_1]$, $B_1 \to A \notin [(\Psi * A) * B_2]$, and $B_2 \to \neg A \in [\Psi * C]$,
     then $B_2 \to \neg A \in [(\Psi * C) * B_1 \veebar B_2]$.

We have already noted that $(\alpha1^*_\preccurlyeq)$ is equivalent to the conjunction of $(\beta1^*_\preccurlyeq)$ and $(\beta3^*_\preccurlyeq)$, while $(\alpha2^*_\preccurlyeq)$ is equivalent to the conjunction of $(\beta2^*_\preccurlyeq)$ and $(\beta4^*_\preccurlyeq)$. The syntactic counterparts of $(\beta1^*_\preccurlyeq)$ and $(\beta2^*_\preccurlyeq)$ were provided in Proposition 5. Parts (a) and (b) of Proposition 11 thus allow us to infer:

Corollary 2. (a) *Given AGM,* $(\alpha1^*_\preccurlyeq)$ *is equivalent to* $(\alpha1^*) := (\beta1^*)$ *&* $(\beta3^*)$.
(b) *Given AGM and* $(C4^*_\preccurlyeq)$, $(\alpha2^*_\preccurlyeq)$ *is equivalent to* $(\alpha2^*) := (\beta2^*)$ *&* $(\beta4^*)$.

Since we already have the syntactic counterpart of $(P^*_\preccurlyeq)$, in view of Theorem 2, part (c) of Proposition 11 completes a second syntactic characterisation of the POI family.

Corollary 3. $*$ *is a POI revision operator iff it satisfies AGM,* (Eq^*), $(C1^*)$, $(C2^*)$, (P^*), $(\alpha1^*)$, $(\alpha2^*)$, *and* $(\alpha3^*)$.

This one-to-one correspondence between semantic and syntactic principles, however, comes at a cost, since we note that $(\alpha1^*)$–$(\alpha3^*)$ are clearly much harder to interpret than $(\Omega1^*)$–$(\Omega3^*)$.

9. POI assignments, overruling and reductionism

In Subsection 2.3, we presented a recent counterexample to the identification of states with TPO's. It suggested that the relations of overruling, which are

28

crucial to two step revision, could not be represented in a structure as simple as a TPO. Indeed, two cases were described, involving respective states that (i) were associated with identical TPO's but (ii) differed as to whether a particular pair of sentences stood in a relation of strict overruling (Example 1, in which B strictly overruled A) or in no relation of overruling at all (Example 2, in which B did not overrule A).

The additional expressive power of POI assignments enables us to adequately model the contrasting situations. The states involved are plausibly associated with the distinct POI assignments depicted in Figure 9, which remain both faithful to the TPO depicted in Figure 1, page 10.
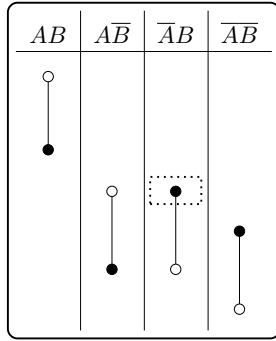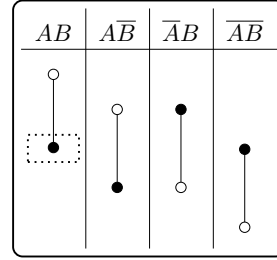
Example 1: $\neg A \in [(\Psi * A) * B]$ Example 2: $A \in [(\Psi * A) * B]$

Figure 9: Representation of POI assignments respectively associated with Example 1 and Example 2, again assuming a language with atomic sentences $\{A, B\}$. Dotted boxes enclose the minimal worlds in $[\![B]\!]$ after revision by A.

It is easy to see that, *in the particular situation at hand*, the distinction between B's strictly overruling A and its not overruling A is down to the following fact: In the POI assignment corresponding to the former case, but not in the one corresponding to the latter, the negative counterpart of the world in $[\![\neg A \wedge B]\!]$ is strictly lower in the ordering than the positive counterpart of the world in $[\![A \wedge B]\!]$. It is therefore natural to ask how the overruling relations are

29

represented in the more general case.

For *non-prioritised* POI revision, the situation is straightforward and was worked out by Booth & Meyer [12]. They first defined two strict partial orders based on the POI construction:

Definition 9. *(i)* $x \ll_\Psi y$ *if* $x^- \leq_\Psi y^+$ *and (ii)* $x \lll_\Psi y$ *if* $x^- <_\Psi y^+$

With this in hand they proved (see their Proposition 8): (a) $A \notin [(\Psi \circ A) * B]$ iff $\min(\ll_\Psi, [\![B]\!]) \subseteq [\![\neg A]\!]$ and (b) $\neg A \in [(\Psi \circ A) * B]$ iff $\min(\lll_\Psi, [\![B]\!]) \subseteq [\![\neg A]\!]$.

In the *prioritised* case, the naturalisation step complicates matters a little. Here, the translation of the overrules relations into $\ll_\Psi$ and $\lll_\Psi$ turns out to be the following:

Proposition 12. *If $*$ is a POI operator, then:*

(a) $A \notin [(\Psi * A) * B]$ *iff* $\min(\ll_\Psi, [\![B]\!]) \subseteq [\![\neg A]\!]$ *and* $\neg B \in [\Psi * A]$*, and*

(b) $\neg A \in [(\Psi * A) * B]$ *iff* $\min(\lll_\Psi, [\![B]\!]) \subseteq [\![\neg A]\!]$ *and* $\neg B \in [\Psi * A]$*.*

As was pointed out in [12], lexicographic revision corresponds to the case in which $\ll_\Psi = \lll_\Psi = \emptyset$, while Papini's reverse lexicographic revision corresponds to the case in which $\ll_\Psi = \lll_\Psi = \prec_\Psi$. Thus, from the above result we see that, for both lexicographic and restrained operators, the overrules and strictly overrules relations collapse into the same relation. Furthermore, for lexicographic revision, we have that B overrules A iff $A \wedge B$ is inconsistent (cf. the postulate (Rec*) in Theorem 1), while, for restrained revision, B overrules A iff both $\neg A \in [\Psi * B]$ and $\neg B \in [\Psi * A]$ (i.e., iff A and B *counteract*, to use the terminology from [7]). Clearly, in both cases, the overrules relation is symmetric, and so unrestricted versions of ($\Omega 1^*$)–($\Omega 3^*$) hold for these two sets of operators.

10. Iterated versions of AGM era postulates

In this final section of the paper, we investigate various further properties of POI revision operators, discussing in the process an interesting issue that has somewhat been neglected in the literature: the extension, to the iterated case, of the various AGM era postulates for revision.

10.1. Some postulates that are sound

In Section 7, we briefly noted that $(\beta 1_{\preccurlyeq}^*)$ and $(\beta 2_{\preccurlyeq}^*)$ could each be reformulated as the conjunction of a pair of principles $((\gamma 1_{\preccurlyeq}^*)$ and $(\gamma 3_{\preccurlyeq}^*)$, regarding the former, and $(\gamma 2_{\preccurlyeq}^*)$ and $(\gamma 4_{\preccurlyeq}^*)$, regarding the latter). We showed that these principles, which had not been discussed in the literature to date, are not generally satisfied by admissible revision operators. It turns out, furthermore, that they are particularly noteworthy, since we can show that, in various combinations, they enable us to recover iterated generalisations of the following strong AGM postulates for revision and related well-known principles:

$(K7^*)$ $\quad [\Psi * A \wedge C] \subseteq \mathrm{Cn}([\Psi * A] \cup \{C\})$

(DR^*) $\quad [\Psi * A \vee C] \subseteq [\Psi * A] \cup [\Psi * C]$

(DO^*) $\quad [\Psi * A] \cap [\Psi * C] \subseteq [\Psi * A \vee C]$

(DI^*) $\quad$ If $\neg A \notin [\Psi * A \vee C]$, then $[\Psi * A \vee C] \subseteq [\Psi * A]$

$(K7^*)$, which was introduced in Section 2, is one of the two 'supplementary' AGM postulates and is also known as 'Superexpansion'. 'DR', 'DO' and 'DI' respectively abbreviate 'Disjunctive Rationality', 'Disjunctive Overlap' and 'Disjunctive Inclusion'. As is well known in the literature, given the other AGM postulates, (DR^*) is a consequence of the second supplementary postulate $(K8^*)$, aka 'Subexpansion', while (DO^*) is equivalent to $(K7^*)$ and (DI^*) to $(K8^*)$, which was also introduced in Section 2.

The iterated generalisations that we recover are obtained by replacing all mentions of the belief states in the principles above by that of their corresponding revisions by a common sentence B and making some minor adjustments. In each case, assuming Success and $[\Psi * \top] = [\Psi]$, setting $B = \top$ enables us to recover the non-iterated counterpart. We have:

$(iK7^*)$ $\quad [(\Psi * A \wedge C) * B] \subseteq \mathrm{Cn}([(\Psi * A) * B] \cup \{A \wedge C\})$

(iDR^*) $\quad [(\Psi * A \vee C) * B] \subseteq [(\Psi * A) * B] \cup [(\Psi * C) * B]$

(iDO^*) $\quad [(\Psi * A) * B] \cap [(\Psi * C) * B] \subseteq [(\Psi * A \vee C) * B]$

(iDI*) If $\neg A \notin [(\Psi * A \vee C) * B]$, then $[(\Psi * A \vee C) * B] \subseteq [(\Psi * A) * B]$

Although (iK7*) and (iDI*) are, to the best of our knowledge, new to the literature, we note that (iDR*) and (iDO*) were already discussed and endorsed by Schlechta *et al* [20]. Our results are the following.

Proposition 13. *In the presence of AGM,* (C1$^*_{\preccurlyeq}$) *and* (C2$^*_{\preccurlyeq}$), **(a)** ($\gamma 1^*_{\preccurlyeq}$) *and* ($\gamma 4^*_{\preccurlyeq}$) *jointly entail* (iDO*) *and* **(b)** ($\gamma 2^*_{\preccurlyeq}$) *and* ($\gamma 3^*_{\preccurlyeq}$) *jointly entail* (iDR*).

Proposition 14. *Given AGM and* (C1$^*_{\preccurlyeq}$), **(a)** ($\gamma 1^*_{\preccurlyeq}$) *is equivalent to* (iDI*) *and* **(b)** ($\gamma 2^*_{\preccurlyeq}$) *is equivalent to* (iK7*).

10.2. Some postulates that are not *sound*

We have not recovered the iterated version of Subexpansion, aka (K8*), whose formulation we recall here

(K8*) If $\neg C \notin [\Psi * A]$, then $\mathrm{Cn}([\Psi * A] \cup \{C\}) \subseteq [\Psi * A \wedge C]$

and which is given by:

(iK8*) If $\neg(A \wedge C) \notin [(\Psi * A) * B]$, then $\mathrm{Cn}([(\Psi * A) * B] \cup \{A \wedge C\})$
$\subseteq [(\Psi * A \wedge C) * B]$

For this, we consider the following rather strong principle:

(sP$^*_{\preccurlyeq}$) If $x \in [\![A]\!]$, $y \in [\![\neg A]\!]$ and $x \preccurlyeq_{\Psi * A \vee C} y$, then $x \prec_{\Psi * A} y$

This principle strengthens the conjunction of ($\gamma 1^*_{\preccurlyeq}$) and ($\gamma 2^*_{\preccurlyeq}$) in much the same way that (P$^*_{\preccurlyeq}$) strengthens the conjunction of (C3$^*_{\preccurlyeq}$) and (C4$^*_{\preccurlyeq}$) (which, recall, are respective weakenings of ($\gamma 1^*_{\preccurlyeq}$) and ($\gamma 2^*_{\preccurlyeq}$)). Taken contrapositively, the principle inherits the weak antecedent of ($\gamma 1^*_{\preccurlyeq}$) but the strong consequent of ($\gamma 2^*_{\preccurlyeq}$). Given $\preccurlyeq_{\Psi * \top} = \preccurlyeq_{\Psi}$, which follows from (C1$^*_{\preccurlyeq}$), (P$^*_{\preccurlyeq}$) is recovered as the special case of (sP$^*_{\preccurlyeq}$) in which $C = \neg A$. We now note:

Proposition 15. (iK8*) *is equivalent to* (sP$^*_{\preccurlyeq}$), *given AGM and* (C1$^*_{\preccurlyeq}$).

Where does our POI family stand with respect to this principle? Well, we can establish the following:

Proposition 16. $(\mathrm{sP}^*_{\preccurlyeq})$ *is satisfied by both lexicographic and restrained revision operators.*

Since lexicographic and restrained revision operators satisfy $(\mathrm{s}\gamma 1^*_{\preccurlyeq})$, this establishes, that they satisfy $(\mathrm{iK8}^*)$. This is interesting, since it shows, not only that the principle is consistent with our previous constraints, but that adding it to these does not yield the kind of 'reductionism' that has been argued to be objectionable. However, it remains the case that

Proposition 17. $(\mathrm{sP}^*_{\preccurlyeq})$ *is not generally satisfied by POI revision operators.*

In fact, a weaker property than this one fails to hold across the family. Indeed, $(\mathrm{sP}^*_{\preccurlyeq})$ generalises the following *Separation* property, discussed by Booth & Meyer [7] under the name of 'UR', which is the special case of $(\mathrm{sP}^*_{\preccurlyeq})$ in which $C = A$:

$(\mathrm{Sep}^*_{\preccurlyeq})$ If $x \in [\![A]\!]$ and $y \in [\![\neg A]\!]$, then $x \prec_{\Psi * A} y$ or $y \prec_{\Psi * A} x$

This condition can be captured by a '*Non-Flush*' constraint on the POI assignment, which states that it is never the case that two intervals line up flush, in the sense that $x^+ \sim_\Psi y^-$. This condition is not satisfied in general by POI assignments. Indeed, consider the following POI assignment to $W = \{x, y, z\}$: $x^+ <_\Psi y^+ <_\Psi x^- <_\Psi y^- \sim_\Psi z^+ <_\Psi z^-$. Non-Flush fails, with the result that so too does $(\mathrm{Sep}^*_{\preccurlyeq})$ and hence $(\mathrm{sP}^*_{\preccurlyeq})$, since $y \sim_{\Psi * x \vee z} z$. This establishes Proposition 17.

At this point, a natural question arises: Why has the narrower family of POI revision operators satisfying $(\mathrm{Sep}^*_{\preccurlyeq})$, or indeed, $(\mathrm{sP}^*_{\preccurlyeq})$, not made a more central appearance in the present paper? The answer to this is that $(\mathrm{Sep}^*_{\preccurlyeq})$ remains in our view an extremely strong property. This becomes most apparent when one considers its syntactic counterpart:

(Sep^*) Either $\neg A \in [(\Psi * A) * B]$ or $A \in [(\Psi * A) * B]$

33

This principle states that, once one has revised one's beliefs by a certain sentence, one will remain opinionated as to whether or not that sentence is true upon any further single revision. But this seems too strong: let A be any sentence and B be the sentence 'The Oracle says that it might not be the case that A'. Plausibly $A, \neg A \notin [(\Psi * A) * B]$.

Also of interest is the iterated version of 'Disjunctive Factoring', which is equivalent to the conjunction of $(K7^*)$ and $(K8^*)$, in the presence of the other AGM postulates:[14]

(DF*) (i)   If $\neg C \in [\Psi * A \vee C]$, then $[\Psi * A \vee C] = [\Psi * A]$

(ii) If $\neg A, \neg C \notin [\Psi * A \vee C]$, then $[\Psi * A \vee C] = [\Psi * A] \cap [\Psi * C]$

(iii) If $\neg A \in [\Psi * A \vee C]$, then $[\Psi * A \vee C] = [\Psi * C]$

The iterated version is given by:

(iDF*) (i)   If $\neg C \in [(\Psi * A \vee C) * B]$, then $[(\Psi * A \vee C) * B]$
$= [(\Psi * A) * B]$

(ii) If $\neg A, \neg C \notin [(\Psi * A \vee C) * B]$, then $[(\Psi * A \vee C) * B]$
$= [(\Psi * A) * B] \cap [(\Psi * C) * B]$

(iii) If $\neg A \in [(\Psi * A \vee C) * B]$, then $[(\Psi * A \vee C) * B]$
$= [(\Psi * C) * B]$

(iDF*)(ii) is entailed by the combination of (iDO*), for the right-to-left direction, and (iDI*), for the left-to-right direction, both of which we have established to be sound for POI revision operators. Regarding (iDF*)(i) (a similar result holding, by symmetry, for (iDF*)(iii)):

Proposition 18. *The semantic counterparts of the right-to-left and left-to-right directions of* (iDF*)(i) *are respectively:*

---

[14]This condition is typically stated in weaker terms, as: $[\Psi * A \vee C]$ is equal to either $[\Psi * A]$, $[\Psi * C]$, or $[\Psi * A] \cap [\Psi * C]$. However, the equivalence that is proven is in fact with the stronger principle. See [29, Proposition 3.16], where the proof is credited to Hans Rott.

$(\gamma 5^*_{\preccurlyeq})$ If $x, y \in [\![\neg A]\!]$ and $y \preccurlyeq_{\Psi * A \vee C} x$, then $y \preccurlyeq_{\Psi * C} x$

$(\gamma 6^*_{\preccurlyeq})$ If $y \in [\![\neg A]\!]$, and $y \prec_{\Psi * A \vee C} x$, then $y \prec_{\Psi * C} x$.

We note that $(\gamma 6^*_{\preccurlyeq})$, in conjunction with $(\gamma 2^*_{\preccurlyeq})$, obviously gives us $(s\beta 2^*_{\preccurlyeq})$. However, due to the requirement that $x \in [\![\neg A]\!]$ in the antecedent of $(\gamma 5^*_{\preccurlyeq})$, the latter does not give us $(s\beta 1^*_{\preccurlyeq})$, in conjunction with $(\gamma 1^*_{\preccurlyeq})$.

Where do these principles stand in relation to our family of operators? The answer is the following:

Proposition 19. *Neither $(\gamma 5^*_{\preccurlyeq})$ nor $(\gamma 6^*_{\preccurlyeq})$ are generally satisfied by POI revision operators.*

However:

Proposition 20. *Both $(\gamma 5^*_{\preccurlyeq})$ and $(\gamma 6^*_{\preccurlyeq})$ are satisfied by lexicographic revision operators.*

This establishes that (iDF^*) is satisfied by lexicographic revision operators, since we have individually shown that they satisfy all the component principles.
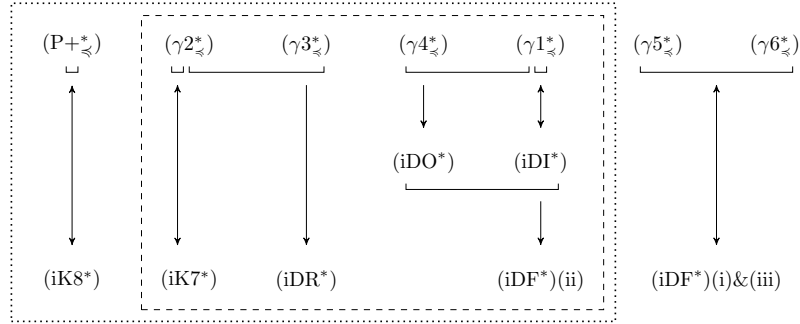
The relations between the semantic and syntactic versions of the postulates discussed in this section are summarised in Figure 10.

11. Conclusions and further work

This paper has investigated a significant, yet comparatively restrained, strengthening of the seminal framework introduced two decades ago by Darwiche and Pearl. Unlike the majority of existing models of iterated revision, the proposal falls short of identifying belief states with simple total preorders over worlds. Indeed, it incorporates further structure into these, in the form of proper ordinal intervals.[15] This is achieved by combining Booth & Meyer's

---

[15]We are not the only ones to have proposed an enrichment of belief states beyond mere TPO's or equivalent structures. One well-known case in point is Spohn's identification of

$(\text{P}+^*_{\preccurlyeq})$ $(\gamma 2^*_{\preccurlyeq})$ $(\gamma 3^*_{\preccurlyeq})$ $(\gamma 4^*_{\preccurlyeq})$ $(\gamma 1^*_{\preccurlyeq})$ $(\gamma 5^*_{\preccurlyeq})$ $(\gamma 6^*_{\preccurlyeq})$

(iDO^*) (iDI^*)

(iK8^*) (iK7^*) (iDR^*) $(\text{iDF}^*)(\text{ii})$ $(\text{iDF}^*)(\text{i})\&(\text{iii})$

Figure 10: Relations between the semantic and syntactic versions of the iterated AGM-era postulates discussed. The dashed box encloses those postulates that are sound for POI operators. The dotted box encloses those that are sound for both lexicographic and restrained revision operators. All postulates are sound for lexicographic revision operators. Note that certain of the implications only hold given one or more of $(\text{C1}^*_{\preccurlyeq})$ and $(\text{C2}^*_{\preccurlyeq})$.

framework for non-prioritised revision with a 'naturalisation' step, in a move that bears some similarities to the definition, via the Levi Identity, of single-step revision as a contraction followed by an expansion. The resulting family of POI revision operators, which is a sub-family of the so-called 'admissible' family, has been characterised both semantically and syntactically. It has also been shown that POI revision operators are distinctive, within the class of admissible ones, in satisfying iterated counterparts of many (albeit not all) classic AGM era postulates.

---

states with '*ranking functions*', aka 'OCFs' [30, 31]. However, Spohn does not acknowledge the concept of revision simpliciter that we are studying. Rather, he considers a parameterised *family* of revision-like functions. This additional parameter very much complicates the translation of our principles into his framework. In a rather different vein, Konieczny & Pérez [32] identify states with *histories of input sentences* (see also [33] and [34] for related work). However, it turns out that if (C2*)–a feature of POI revision–is imposed, then the class of operators that they study narrows down to lexicographic revision alone.

In future work, we first plan to consider the consequences of relaxing (P^*). This condition fails for a more general family of what one could call '*basic ordinal interval (BOI)*' revision operators. These operators, which include natural revision operators, are naturalisations of non-prioritised operators based on ordinal interval assignments that satisfy, not (≤ 2), but the weaker requirement that $x^+ \leq x^-$. As it turns out, our proof of the soundness of $(\alpha 1^*_{\preccurlyeq})$–$(\alpha 3^*_{\preccurlyeq})$ with respect to POI operators carries over here, leaving us in a strong position to provide a characterisation for this more general family.

Secondly, as Figure 4 reminds us, the constraints that we have discussed impose few constraints on the result of more than two iterations of the revision operation. While the structure associated with belief states currently determines the posterior *TPO*, nothing has been said regarding the nature of the posterior *POI assignment*. One possible question to investigate in relation to this would be that of the extent to which the DP postulates, and the new principles introduced here, could be adapted to the relations $\ll$ and $\lll$ introduced in Definition 9.

Finally, we have not provided a discussion of any complexity considerations. Relevant problems whose computational difficulty one might want to to consider include the so-called 'inference' problem of deciding, for a given state Ψ and sentences A, B and C, whether or not $C \in [(\Psi * A) * B]$, where $*$ is a POI revision operator.[16]

Acknowledgments

---

[16]See for instance [35] for a summary of the early literature on the analogous problem that arises in the context of single-step revision. [36] discusses the issue of inference, alongside a number of others, in relation to a class of iterated revision operators that partly overlaps with ours, since it includes lexicographic revision.

Appendix A: A sample 'concrete' POI revision operator

We use an idea from Booth & Meyer [12], that bears some superficial similarities to Spohn's ranking theory [30, 31]. We identify the set of possible belief states with the set of all functions $p : W \to \mathbb{R}$. For each p, we have an associated TPO $\preccurlyeq_p$ over W, given by setting $x \preccurlyeq_p y$ iff $p(x) \leq p(y)$. For each sentence $B \in L$ we define $p(B) = \min\{p(y) \mid y \in [\![B]\!]\}$.

We have a parameter $a > 0$ as fixed and given. Then each p can be extended to a function on *signed* worlds by setting, for each $x \in W$, $p(x^+) = p(x)$ and $p(x^-) = p(x) + a$, which yields a POI assignment $\leq_p$ given by $x^\epsilon \leq_p y^\delta$ iff $p(x^\epsilon) \leq p(y^\delta)$. In other words, we assign to each x the real interval $(p(x), p(x) + a)$.

We define the revision $p *_{a,b} A$ of p by A as a composition of two operators $\circ_a$ and $\oplus_b$, i.e., $p *_{a,b} A = (p \circ_a A) \oplus_b A$. The first $\circ_a$ is a non-prioritised revision, as defined previously, and the second $\oplus_b$ is a natural revision operator, parameterised by some fixed $b > 0$. More precisely, $p \circ_a A$ is given by setting, for each $x \in W$,

$$[p \circ_a A](x) = \begin{cases} p(x) & \text{if } x \in [\![A]\!] \\ p(x) + a & \text{if } x \in [\![\neg A]\!] \end{cases}$$

In other words, all countermodels of the new information get their p-value increased by a. Note that the interval associated to x will stay unchanged if $x \in [\![A]\!]$, while if $x \in [\![\neg A]\!]$ then the whole interval gets shifted by amount a to $(p(x) + a, p(x) + 2a)$.

Then $p \oplus_b A$ is defined by setting $p \oplus_b A = p$ if $\min(\preccurlyeq_p, W) \subseteq [\![A]\!]$. Otherwise we set

$$[p \oplus_b A](x) = \begin{cases} p(\top) - b & \text{if } x \in \min(\preccurlyeq_p, [\![A]\!]) \\ p(x) & \text{otherwise} \end{cases}$$

We provide an example of $*_{a,b}$ in action (assuming $a = b = 2$) in Figure 11. We know that $\circ_a$ is a non-prioritised POI revision operator from [12] and it is easy to see that $\mathbb{N}(*_{a,b}, \circ_a)$. Thus, $*_{a,b}$ so defined is a POI revision operator and
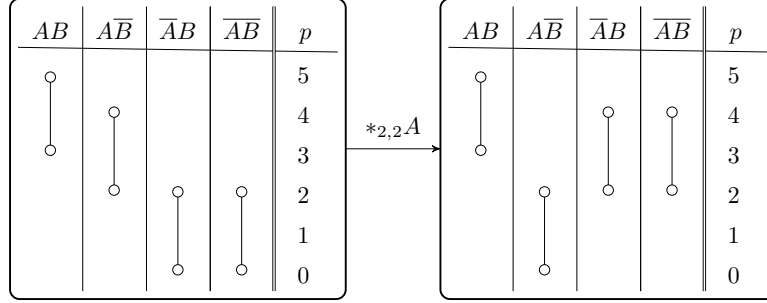
Figure 11: Interval representation of the revision of a state by A, with parameters $a = b = 2$. The lower endpoint of each interval indicates the p-value $p(x)$ of a given world x and its upper endpoint the value $p(x) + a = p(x) + 2$.

satisfies all the relevant postulates discussed in the previous sections.

Appendix B: Proofs

Proposition 1. $\mathbb{N}(*, \circ)$ *iff, for all* $A \in L$, $\preccurlyeq_{\Psi * A}$ *minimises the symmetric difference distance* d_S *to* $\preccurlyeq_{\Psi \circ A}$, *subject to the constraints of AGM.*

Proof: Let $*$ satisfy AGM. Then, we will have $\min(\preccurlyeq_{\Psi * A}, W) = \min(\preccurlyeq_{\Psi}, [\![A]\!])$. We also know that $\min(\preccurlyeq_{\Psi}, [\![A]\!]) = \min(\preccurlyeq_{\Psi \circ A}, [\![A]\!])$. Indeed, since (i) for all $x \in [\![A]\!]$, $r_A(x) = x^+$, (ii) $x^+ \leq_{\Psi} y^+$ iff $x \preccurlyeq_{\Psi} y$, and (iii) $x \preccurlyeq_{\Psi \circ A} y$ iff $r_A(x) \leq_{\Psi} r_A(y)$, we have, for all $x, y \in [\![A]\!]$, $x \preccurlyeq_{\Psi \circ A} y$ iff $x \preccurlyeq_{\Psi} y$. Hence $\min(\preccurlyeq_{\Psi * A}, W) = \min(\preccurlyeq_{\Psi \circ A}, [\![A]\!])$. By the definition of natural revision, $x \preccurlyeq_{(\Psi \circ A) \boxplus A} y$ iff: (i) $x \in \min(\preccurlyeq_{\Psi \circ A}, [\![A]\!])$, or (ii) $x, y \notin \min(\preccurlyeq_{\Psi \circ A}, [\![A]\!])$ and $x \preccurlyeq_{\Psi \circ A} y$. From this we can then conclude, as required, that $\preccurlyeq_{\Psi * A}$ minimises the symmetric difference distance d_S to $\preccurlyeq_{\Psi \circ A}$ iff it is equal to $\preccurlyeq_{(\Psi \circ A) \boxplus A}$, i.e. iff $\mathbb{N}(*, \circ)$. $\qquad \square$

Proposition 2. *For any functions* $\circ$ *and* $*$ *from state-sentence pairs to states, such that* $\mathbb{N}(*, \circ)$, *if* $\circ$ *satisfies* (Eq$_{\preccurlyeq}^{\circ}$), (C1$_{\preccurlyeq}^{\circ}$), (C2$_{\preccurlyeq}^{\circ}$) *and* (P$_{\preccurlyeq}^{\circ}$), *then* $*$ *will satisfy* (Eq$_{\preccurlyeq}^{*}$), (C1$_{\preccurlyeq}^{*}$), (C2$_{\preccurlyeq}^{*}$) *and* (P$_{\preccurlyeq}^{*}$).

Proof: Recall that $\mathbb{N}(*, \circ)$ iff: $x \preccurlyeq_{\Psi * A} y$ iff either (i) $x \in \min(\preccurlyeq_{\Psi \circ A}, [\![A]\!])$, or (ii) $x, y \notin \min(\preccurlyeq_{\Psi \circ A}, [\![A]\!])$ and $x \preccurlyeq_{\Psi \circ A} y$.

(i) **Preservation of** $(\mathrm{C1}^*_{\preccurlyeq})$: Assume that $x, y \in [\![A]\!]$ and that $x \preccurlyeq_{\Psi \circ A} y$ iff $x \preccurlyeq_{\Psi} y$. We show that $x \preccurlyeq_{\Psi \circ A} y$ iff $x \preccurlyeq_{\Psi * A} y$.

From $x \preccurlyeq_{\Psi * A} y$ to $x \preccurlyeq_{\Psi \circ A} y$: Trivial.

From $x \preccurlyeq_{\Psi \circ A} y$ to $x \preccurlyeq_{\Psi * A} y$: Assume $x \preccurlyeq_{\Psi \circ A} y$. We just need to show that, if $x \notin \min(\preccurlyeq_{\Psi \circ A}, [\![A]\!])$, then $y \notin \min(\preccurlyeq_{\Psi \circ A}, [\![A]\!])$. This follows immediately from $x \preccurlyeq_{\Psi \circ A} y$ and $x \in [\![A]\!]$.

(ii) **Preservation of** $(\mathrm{C2}^*_{\preccurlyeq})$: Assume that $x, y \in [\![A]\!]$ and that $x \preccurlyeq_{\Psi \circ A} y$ iff $x \preccurlyeq_{\Psi} y$. We show that $x \preccurlyeq_{\Psi \circ A} y$ iff $x \preccurlyeq_{\Psi * A} y$.

From $x \preccurlyeq_{\Psi * A} y$ to $x \preccurlyeq_{\Psi \circ A} y$: Trivial.

From $x \preccurlyeq_{\Psi \circ A} y$ to $x \preccurlyeq_{\Psi * A} y$: Assume $x \preccurlyeq_{\Psi \circ A} y$. We just need to show that, if $x \notin \min(\preccurlyeq_{\Psi \circ A}, [\![A]\!])$, then $y \notin \min(\preccurlyeq_{\Psi \circ A}, [\![A]\!])$. This follows immediately from $y \in [\![\neg A]\!]$.

(iii) **Preservation of** $(\mathrm{P}^*_{\preccurlyeq})$: Assume that $x \in [\![A]\!]$, $y \in [\![\neg A]\!]$ and that, if $x \preccurlyeq_{\Psi} y$ then $x \prec_{\Psi \circ A} y$. We show that, if $x \prec_{\Psi \circ A} y$, then $x \prec_{\Psi * A} y$. From the definition of naturalisation, we have: if $\mathbb{N}(*, \circ)$, then $x \prec_{\Psi * A} y$ iff either (i) $x \in \min(\preccurlyeq_{\Psi \circ A}, [\![A]\!])$ and $y \notin \min(\preccurlyeq_{\Psi \circ A}, [\![A]\!])$, or (ii) $x, y \notin \min(\preccurlyeq_{\Psi \circ A}, [\![A]\!])$ and $x \prec_{\Psi \circ A} y$. So, given $x \prec_{\Psi \circ A} y$, we just need to show that, if $x \in \min(\preccurlyeq_{\Psi \circ A}, [\![A]\!])$, then $y \notin \min(\preccurlyeq_{\Psi \circ A}, [\![A]\!])$. This follows immediately from $y \in [\![\neg A]\!]$.

(iv) **Preservation of** $(\mathrm{Eq}^*_{\preccurlyeq})$: Assume $A \equiv B$. We want to show $\preccurlyeq_{\Psi * A} = \preccurlyeq_{\Psi * B}$. So assume for contradiction that there exist x and y such that $x \preccurlyeq_{\Psi * A} y$ but $y \prec_{\Psi * B} x$ (the other case is analogous).

From $x \preccurlyeq_{\Psi * A} y$ we have: (1) $x \in \min(\preccurlyeq_{\Psi \circ A}, [\![A]\!])$, or (2) $x, y \notin \min(\preccurlyeq_{\Psi \circ A}, [\![A]\!])$ and $x \preccurlyeq_{\Psi \circ A} y$.

From $y \prec_{\Psi * A} x$, we obtain: (3) $y \in \min(\preccurlyeq_{\Psi \circ B}, \llbracket B \rrbracket)$ and $x \notin \min(\preccurlyeq_{\Psi \circ B}, \llbracket B \rrbracket)$, or (4) $x, y \notin \min(\preccurlyeq_{\Psi \circ B}, \llbracket B \rrbracket)$ and $y \prec_{\Psi \circ B} y$.

From $A \equiv B$ and the fact that $\circ$ satisfies $(\mathrm{Eq}^*_{\preccurlyeq})$, we have $\preccurlyeq_{\Psi \circ A} = \preccurlyeq_{\Psi \circ B}$ and $\llbracket A \rrbracket = \llbracket B \rrbracket$. Given this, it is easy to see that neither (1) nor (2) is consistent with either (3) or (4). $\qquad\square$

Proposition 4. *Let Ψ and Ψ' be two belief states and $*$ a POI revision operator. Then the following two statements are equivalent:*

(1) $\forall A \in L, \preccurlyeq_{\Psi * A} = \preccurlyeq_{\Psi' * A}$

(2) (a) $\preccurlyeq_{\Psi'} = \preccurlyeq_{\Psi} = \preccurlyeq$, and (b) $\leq_{\Psi'}$ and $\leq_{\Psi}$ agree on $\{x^-, y^+\}$ for all $x, y \in W$ such that

(i) $x \prec y$ and

(ii) there exists $z \in W$ such that $z \neq x$ and $z \prec y$.

Proof: For the proof, we shall use the syntactic counterpart of (1), namely: $\forall A, B \in L, [(\Psi * A) * B] = [(\Psi' * A) * B)]$. We note, in relation to part (b), that Proposition 2 above entitles us to help ourselves to the postulates $(\mathrm{C1}^*_{\preccurlyeq})$ to $(\mathrm{C4}^*_{\preccurlyeq})$.

(a) **Left-to-right direction:** We prove the contrapositive, deriving, from the negation of (2), the claim that there exist $A, B \in L$ such that $[(\Psi * A) * B] \neq [(\Psi' * A) * B)]$.

So assume that either (a) $\preccurlyeq_{\Psi'} \neq \preccurlyeq_{\Psi}$ or (b) $\preccurlyeq_{\Psi'} = \preccurlyeq_{\Psi} = \preccurlyeq$ and there exist $x, y \in W$ such that (i) $x \prec y$, (ii) there exists $z \in W$ such that $z \neq x$ and $z \prec y$, and (iii) $\leq_{\Psi'}$ and $\leq_{\Psi}$ disagree on $\{x^-, y^+\}$.

If (a), then it is easy to see that there exists $A \in L$ such that $[\Psi * A] \neq [\Psi' * A]$. Hence $[(\Psi * A) * \top] \neq [(\Psi' * A) * \top)]$ and we are done.

So assume (b). Consider now the three possibilities regarding the manner in which $\leq_{\Psi}$ relates x^- and y^+: (1) $x^- <_{\Psi} y^+$, (2) $x^- \doteq_{\Psi} y^+$ (i.e.,

both $x^- \leq_\Psi y^+$ and $y^+ \leq_\Psi x^-$) and (3) $y^+ <_\Psi x^-$. Given (i) and (ii), each of these cases will yield a different result for a revision of the belief state by $y \vee z$ and then by $x \vee y$, namely:

(1) $x \in [(\Psi * y \vee z) * x \vee y]$

(2) $x, y \notin [(\Psi * y \vee z) * x \vee y]$

(3) $y \in [(\Psi * y \vee z) * x \vee y]$

This situation is analogous for $\leq_{\Psi'}$. Hence any disagreement between $\leq_{\Psi'}$ and $\leq_\Psi$ on $\{x^-, y^+\}$, will yield a difference in outcomes after revision of Ψ and Ψ', respectively, by $y \vee z$ and then by $x \vee y$.

(b) **Right-to-left direction:** Assume that, for some $A, B \in L$, $[(\Psi * A) * B] \neq [(\Psi' * A) * B)]$, i.e. $\min(\preccurlyeq_{\Psi*A}, [\![B]\!]) \neq \min(\preccurlyeq_{\Psi'*A}, [\![B]\!])$. We will show that, if (a) $\preccurlyeq_{\Psi'} = \preccurlyeq_\Psi = \preccurlyeq$, then (b) there exist $x, y \in W$ such that (i) $x \prec y$, (ii) there exists $z \in W$ such that $z \neq x$ and $z \prec y$, and (iii) $\leq_{\Psi'}$ and $\leq_\Psi$ disagree on $\{x^-, y^+\}$.

Assume for definiteness that there exists $y \in \min(\preccurlyeq_{\Psi*A}, [\![B]\!])$ such that $y \notin \min(\preccurlyeq_{\Psi'*A}, [\![B]\!])$ (the reasoning from the case in which y is in the second set but not the first is analogous). Then there exists x (in $\min(\preccurlyeq_{\Psi'*A}, [\![B]\!])$) such that $x \prec_{\Psi'*A} y$ but $y \preccurlyeq_{\Psi*A} x$.

Now assume that $\preccurlyeq_{\Psi'} = \preccurlyeq_\Psi = \preccurlyeq$. By the definition of POI assignments, we then have $x \prec_{\Psi'*A} y$ iff

(1) $x \in \min(\preccurlyeq, [\![A]\!])$ and $y \notin \min(\preccurlyeq_\Psi, [\![A]\!])$, or

(2) $x, y \notin \min(\preccurlyeq, [\![A]\!])$ and $r_A(x) <_{\Psi'} r_A(y)$

as well as $y \preccurlyeq_{\Psi*A} x$ iff

(3) $y \in \min(\preccurlyeq, [\![A]\!])$, or

(4) $x, y \notin \min(\preccurlyeq, [\![A]\!])$ and $r_A(y) \leq_\Psi r_A(x)$

But (1) is inconsistent with both (3) and (4) and (3) is inconsistent with (2). This means that we must have (2) and (4).

Since $x \prec_{\Psi' * A} y$ but $y \preccurlyeq_{\Psi * A} x$, by $(C1^*_{\preccurlyeq})$ and $(C2^*_{\preccurlyeq})$, x and y can't both be in $\llbracket A \rrbracket$ or both be in $\llbracket \neg A \rrbracket$.

So assume $x \in \llbracket \neg A \rrbracket$ and $y \in \llbracket A \rrbracket$. By $(C4^*_{\preccurlyeq})$ and $x \prec_{\Psi' * A} y$, we then have $x \prec y$. Given our assumption, $r_A(y) = y^+$ and $r_A(x) = x^-$. So by (2), we have $x^- <_{\Psi'} y^+$ and by (4), we have $y^+ \leq_{\Psi} x^-$. So $\leq_{\Psi'}$ and $\leq_{\Psi}$ disagree on $\{x^-, y^+\}$. We simply now need to establish that there exists $z \neq x$ such that $z \prec y$. We know that, although $y \in \llbracket A \rrbracket$, $y \notin \min(\preccurlyeq, \llbracket A \rrbracket)$. So there exists $z \in \llbracket A \rrbracket$, such that $z \prec y$. Furthermore, since $z \in \llbracket A \rrbracket$ but $x \in \llbracket \neg A \rrbracket$, $z \neq x$.

Assume then that $x \in \llbracket A \rrbracket$ and $y \in \llbracket \neg A \rrbracket$. Given our assumption, $r_A(y) = y^-$ and $r_A(x) = x^+$. So by (2), we have $x^+ <_{\Psi'} y^-$ and by (4), we have $y^- \leq_{\Psi} x^+$. So $\leq_{\Psi'}$ and $\leq_{\Psi}$ disagree on $\{x^-, y^+\}$. Furthermore, since by definition, $y^+ <_{\Psi} y^-$, it follows from $y^- \leq_{\Psi} x^+$ that $y^+ <_{\Psi} x^+$ and therefore $y \prec x$. We simply now need to establish that there exists $z \neq y$ such that $z \prec x$. We know that, although $x \in \llbracket A \rrbracket$, $x \notin \min(\preccurlyeq, \llbracket A \rrbracket)$. So there exists $z \in \llbracket A \rrbracket$, such that $z \prec x$. Furthermore, since $z \in \llbracket A \rrbracket$ but $y \in \llbracket \neg A \rrbracket$, $z \neq y$. $\qquad\square$

Proposition 5. (a) *Given AGM, $(\beta 1^*_{\preccurlyeq})$ is equivalent to:*

$(\beta 1^*)$ *If $A \notin [(\Psi * A) * B]$ and $B \rightarrow \neg A \in [\Psi * C]$, then $A \notin [(\Psi * C) * B]$*

(b) *Given AGM, $(\beta 2^*_{\preccurlyeq})$ is equivalent to:*

$(\beta 2^*)$ *If $\neg A \in [(\Psi * A) * B]$ and $B \rightarrow \neg A \in [\Psi * C]$, then $\neg A \in [(\Psi * C) * B]$*

Proof:

  (a)   **(i) From $(\beta 1^*_{\preccurlyeq})$ to $(\beta 1^*)$:** Assume $A \notin [(\Psi * A) * B]$, $B \rightarrow \neg A \in [\Psi * C]$ and, for contradiction, $A \in [(\Psi * C) * B]$. From $A \in [(\Psi * C) * B]$, it follows that $\min(\preccurlyeq_{\Psi * C}, \llbracket B \rrbracket) \subseteq \llbracket A \rrbracket$. Now consider an arbitrary $x \in \min(\preccurlyeq_{\Psi * C}, \llbracket B \rrbracket)$. Since $x \in \llbracket A \wedge B \rrbracket$, it follows from $\neg(A \wedge B) \in$

$[\Psi * C]$ and Success that $x \notin \min(\preccurlyeq_\Psi, \llbracket C \rrbracket)$. From $A \notin [(\Psi * A) * B]$, there exists y such that $y \in \llbracket \neg A \rrbracket \cap \min(\preccurlyeq_{\Psi*A}, \llbracket B \rrbracket)$. Given $x \in \llbracket B \rrbracket$, we furthermore have $y \preccurlyeq_{\Psi*A} x$. By $(\beta1^*_\preccurlyeq)$, we then recover $y \preccurlyeq_{\Psi*C} x$. Since $y \in \llbracket B \rrbracket$, $x \in \min(\preccurlyeq_{\Psi*C}, \llbracket B \rrbracket)$ and $y \in \llbracket \neg A \rrbracket$, this contradicts $\min(\preccurlyeq_{\Psi*C}, \llbracket B \rrbracket) \subseteq \llbracket A \rrbracket$. Hence $A \notin [(\Psi * C) * B]$, as required.

(ii) From $(\beta1^*)$ to $(\beta1^*_\preccurlyeq)$: Assume $x \notin \min(\preccurlyeq_\Psi, \llbracket C \rrbracket)$, $x \in \llbracket A \rrbracket$, $y \in \llbracket \neg A \rrbracket$ and $y \preccurlyeq_{\Psi*A} x$. Assume for contradiction that $\neg(A \wedge (x \vee y)) \notin [\Psi * C]$. Then there exists w in $\min(\preccurlyeq_\Psi, \llbracket C \rrbracket) \cap \llbracket A \wedge (x \vee y) \rrbracket$. But $\llbracket A \wedge (x \vee y) \rrbracket = \{x\}$, since $x \in \llbracket A \rrbracket$ and $y \in \llbracket \neg A \rrbracket$, and we have assumed $x \notin \min(\preccurlyeq_\Psi, \llbracket C \rrbracket)$. Contradiction. So $\neg(A \wedge (x \vee y)) \in [\Psi * C]$. From $y \in \llbracket \neg A \rrbracket$ and $y \preccurlyeq_{\Psi*A} x$, it follows that $A \notin [(\Psi * A) * x \vee y]$. By $(\beta1^*)$, we then recover $A \notin [(\Psi * C) * x \vee y]$ and hence, since $x \in \llbracket A \rrbracket$ and $y \in \llbracket \neg A \rrbracket$, $y \preccurlyeq_{\Psi*C} x$, as required.

(b) (i) From $(\beta2^*_\preccurlyeq)$ to $(\beta2^*)$: Assume $\neg A \in [(\Psi * A) * B]$, $B \to \neg A \in [\Psi * C]$ and, for contradiction, $\neg A \notin [(\Psi * C) * B]$. From $\neg A \notin [(\Psi * C) * B]$, there exists x such that $x \in \llbracket A \rrbracket \cap \min(\preccurlyeq_{\Psi*C}, \llbracket B \rrbracket)$. Since $x \in \llbracket A \wedge B \rrbracket$, it follows from $\neg(A \wedge B) \in [\Psi * C]$ and Success that $x \notin \min(\preccurlyeq_\Psi, \llbracket C \rrbracket)$. From $\neg A \in [(\Psi * A) * B]$, there exists y such that $y \in \llbracket \neg A \rrbracket \cap \min(\preccurlyeq_{\Psi*A}, \llbracket B \rrbracket)$. Given $x \in \llbracket B \rrbracket$, we furthermore have $y \prec_{\Psi*A} x$. By $(\beta2^*_\preccurlyeq)$, we then recover $y \prec_{\Psi*C} x$. Since $y \in \llbracket B \rrbracket$, this contradicts $x \in \min(\preccurlyeq_{\Psi*C}, \llbracket B \rrbracket)$. Hence $\neg A \in [(\Psi * C) * B]$, as required.

(ii) From $(\beta2^*)$ to $(\beta2^*_\preccurlyeq)$: Assume $x \notin \min(\preccurlyeq_\Psi, \llbracket C \rrbracket)$, $x \in \llbracket A \rrbracket$, $y \in \llbracket \neg A \rrbracket$ and $y \prec_{\Psi*A} x$. Assume for contradiction that $\neg(A \wedge (x \vee y)) \notin [\Psi * C]$. Then there exists w in $\min(\preccurlyeq_\Psi, \llbracket C \rrbracket) \cap \llbracket A \wedge (x \vee y) \rrbracket$. But $\llbracket A \wedge (x \vee y) \rrbracket = \{x\}$, since $x \in \llbracket A \rrbracket$ and $y \in \llbracket \neg A \rrbracket$, and we have assumed $x \notin \min(\preccurlyeq_\Psi, \llbracket C \rrbracket)$. Contradiction. So $\neg(A \wedge (x \vee y)) \in [\Psi * C]$. From $y \in \llbracket \neg A \rrbracket$ and $y \prec_{\Psi*A} x$, it follows that $\neg A \in [(\Psi * A) * x \vee y]$. By $(\beta2^*)$, we then recover $\neg A \in [(\Psi * C) * x \vee y]$

and hence, since $x \in \llbracket A \rrbracket$ and $y \in \llbracket \neg A \rrbracket$, $y \prec_{\Psi * C} x$, as required. $\square$

Proposition 6. (a) *Given* $(\text{C2}^*_{\preccurlyeq})$ *and* $(\text{C4}^*_{\preccurlyeq})$, $(\beta 1^*_{\preccurlyeq})$ *is equivalent to the conjunction of the following two principles:*

$(\gamma 1^*_{\preccurlyeq})$ *If* $x \in \llbracket A \rrbracket$, $y \in \llbracket \neg A \rrbracket$ *and* $y \preccurlyeq_{\Psi * A} x$, *then* $y \preccurlyeq_{\Psi * A \vee C} x$

$(\gamma 3^*_{\preccurlyeq})$ *If* $x \notin \min(\preccurlyeq_{\Psi}, \llbracket C \rrbracket)$, $x \in \llbracket A \vee C \rrbracket$, $y \in \llbracket \neg(A \vee C) \rrbracket$, *and* $y \preccurlyeq_{\Psi * A \vee C} x$, *then* $y \preccurlyeq_{\Psi * C} x$.

(b) *Given* $(\text{C1}^*_{\preccurlyeq})$ *and* $(\text{C3}^*_{\preccurlyeq})$, $(\beta 2^*_{\preccurlyeq})$ *is equivalent to the conjunction of the following two principles:*

$(\gamma 2^*_{\preccurlyeq})$ *If* $x \in \llbracket A \rrbracket$, $y \in \llbracket \neg A \rrbracket$ *and* $y \prec_{\Psi * A} x$, *then* $y \prec_{\Psi * A \vee C} x$

$(\gamma 4^*_{\preccurlyeq})$ *If* $x \notin \min(\preccurlyeq_{\Psi}, \llbracket C \rrbracket)$, $x \in \llbracket A \vee C \rrbracket$, $y \in \llbracket \neg(A \vee C) \rrbracket$ *and* $y \prec_{\Psi * A \vee C} x$, *then* $y \prec_{\Psi * C} x$.

Proof:

    (a)   **(i) From** $(\gamma 1^*_{\preccurlyeq})$ **and** $(\gamma 3^*_{\preccurlyeq})$ **to** $(\beta 1^*_{\preccurlyeq})$**:** From $(\text{C2}^*_{\preccurlyeq})$ and $(\text{C4}^*_{\preccurlyeq})$, we obtain:

        (1)   If $x \notin \min(\preccurlyeq_{\Psi}, \llbracket C \rrbracket)$, $x \in \llbracket A \vee C \rrbracket$, $y \in \llbracket C \rrbracket$ and $y \preccurlyeq_{\Psi * A \vee C} x$, then $y \preccurlyeq_{\Psi * C} x$

        Indeed, from $x \in \llbracket A \vee C \rrbracket$, $y \in \llbracket C \rrbracket \subseteq \llbracket A \vee C \rrbracket$ and $y \preccurlyeq_{\Psi * A \vee C} x$, $(\text{C2}^*_{\preccurlyeq})$ gives us $y \preccurlyeq_{\Psi} x$. Given $y \in \llbracket C \rrbracket$, if we assume $x \in \llbracket C \rrbracket$, then $y \preccurlyeq_{\Psi * C} x$ follows from $(\text{C2}^*_{\preccurlyeq})$. If we assume instead that $x \in \llbracket \neg C \rrbracket$, the same conclusion follows from $(\text{C4}^*_{\preccurlyeq})$. The conjunction of $(\gamma 3^*_{\preccurlyeq})$ and (1) then gives us:

        (2)   If $x \notin \min(\preccurlyeq_{\Psi}, \llbracket C \rrbracket)$, $x \in \llbracket A \vee C \rrbracket$, $y \in \llbracket \neg A \vee C \rrbracket$ and $y \preccurlyeq_{\Psi * A \vee C} x$, then $y \preccurlyeq_{\Psi * C} x$

We now show that (2) and $(\gamma 1_{\preccurlyeq}^{*})$ entail $(\beta 1_{\preccurlyeq}^{*})$: Assume $x \in \llbracket A \rrbracket$, $y \in \llbracket \neg A \rrbracket$, $x \notin \min(\preccurlyeq_{\Psi}, \llbracket C \rrbracket)$ and $y \preccurlyeq_{\Psi * A} x$. From $x \in \llbracket A \rrbracket$, $y \in \llbracket \neg A \rrbracket$, and $y \preccurlyeq_{\Psi * A} x$, $(\gamma 1_{\preccurlyeq}^{*})$ gives us $y \preccurlyeq_{\Psi * A \vee C} x$. From $x \in \llbracket A \rrbracket$ and $y \in \llbracket \neg A \rrbracket$, we recover $x \in \llbracket A \vee C \rrbracket$ and $y \in \llbracket \neg A \vee C \rrbracket$. From $x \notin \min(\preccurlyeq_{\Psi}, \llbracket C \rrbracket)$, $x \in \llbracket A \vee C \rrbracket$, $y \in \llbracket \neg A \vee C \rrbracket$ and $y \preccurlyeq_{\Psi * A \vee C} x$, (2) gives us $y \preccurlyeq_{\Psi * C} x$, as required.

(ii) **From $(\beta 1_{\preccurlyeq}^{*})$ to $(\gamma 1_{\preccurlyeq}^{*})$ and $(\gamma 3_{\preccurlyeq}^{*})$:** To get from $(\beta 1_{\preccurlyeq}^{*})$ to $(\gamma 3_{\preccurlyeq}^{*})$, simply substitute $A \vee C$ for A in $(\beta 1_{\preccurlyeq}^{*})$. To get from $(\beta 1_{\preccurlyeq}^{*})$ to $(\gamma 1_{\preccurlyeq}^{*})$, assume $x \in \llbracket A \rrbracket$, $y \in \llbracket \neg A \rrbracket$ and $y \preccurlyeq_{\Psi * A} x$. From $y \preccurlyeq_{\Psi * A} x$ and $y \in \llbracket \neg A \rrbracket$, it follows by Success that $x \notin \min(\preccurlyeq_{\Psi}, \llbracket A \rrbracket)$ and hence by $x \in \llbracket A \rrbracket$ that $x \notin \min(\preccurlyeq_{\Psi}, \llbracket A \vee C \rrbracket)$. By $(\beta 1_{\preccurlyeq}^{*})$, we then recover $y \preccurlyeq_{\Psi * A \vee C} x$, as required.

(b) (i) **From $(\gamma 2_{\preccurlyeq}^{*})$ and $(\gamma 4_{\preccurlyeq}^{*})$ to $(\beta 2_{\preccurlyeq}^{*})$:** From $(C1_{\preccurlyeq}^{*})$ and $(C3_{\preccurlyeq}^{*})$, we obtain:

(1)   If $x \notin \min(\preccurlyeq_{\Psi}, \llbracket C \rrbracket)$, $x \in \llbracket A \vee C \rrbracket$, $y \in \llbracket C \rrbracket$ and $y \prec_{\Psi * A \vee C} x$, then $y \prec_{\Psi * C} x$

Indeed, from $x \in \llbracket A \vee C \rrbracket$, $y \in \llbracket C \rrbracket \subseteq \llbracket A \vee C \rrbracket$ and $y \prec_{\Psi * A \vee C} x$, $(C1_{\preccurlyeq}^{*})$ gives us $y \prec_{\Psi} x$. Given $y \in \llbracket C \rrbracket$, if we assume $x \in \llbracket C \rrbracket$, then $y \prec_{\Psi * C} x$ follows from $(C1_{\preccurlyeq}^{*})$. If we assume instead that $x \in \llbracket \neg C \rrbracket$, the same conclusion follows from $(C3_{\preccurlyeq}^{*})$. The conjunction of $(\gamma 4_{\preccurlyeq}^{*})$ and (1) then gives us:

(2)   If $x \notin \min(\preccurlyeq_{\Psi}, \llbracket C \rrbracket)$, $x \in \llbracket A \vee C \rrbracket$, $y \in \llbracket \neg A \vee C \rrbracket$ and $y \prec_{\Psi * A \vee C} x$, then $y \prec_{\Psi * C} x$

We now show that (2) and $(\gamma 2_{\preccurlyeq}^{*})$ entail $(\beta 2_{\preccurlyeq}^{*})$: Assume $x \in \llbracket A \rrbracket$, $y \in \llbracket \neg A \rrbracket$, $x \notin \min(\preccurlyeq_{\Psi}, \llbracket C \rrbracket)$ and $y \prec_{\Psi * A} x$. From $x \in \llbracket A \rrbracket$, $y \in \llbracket \neg A \rrbracket$, and $y \prec_{\Psi * A} x$, $(\gamma 2_{\preccurlyeq}^{*})$ gives us $y \prec_{\Psi * A \vee C} x$. From $x \in \llbracket A \rrbracket$ and $y \in \llbracket \neg A \rrbracket$, we recover $x \in \llbracket A \vee C \rrbracket$ and $y \in \llbracket \neg A \vee C \rrbracket$. From $x \notin \min(\preccurlyeq_{\Psi}, \llbracket C \rrbracket)$, $x \in \llbracket A \vee C \rrbracket$, $y \in \llbracket \neg A \vee C \rrbracket$ and $y \prec_{\Psi * A \vee C} x$, (2) gives us $y \prec_{\Psi * C} x$, as required.

(ii) From $(\beta 2^*_{\preccurlyeq})$ **to the conjunction of** $(\gamma 2^*_{\preccurlyeq})$ **and** $(\gamma 4^*_{\preccurlyeq})$**:** To get from $(\beta 2^*_{\preccurlyeq})$ to $(\gamma 4^*_{\preccurlyeq})$, simply substitute $A \vee C$ for A in $(\beta 2^*_{\preccurlyeq})$. To get from $(\beta 2^*_{\preccurlyeq})$ to $(\gamma 2^*_{\preccurlyeq})$, assume $x \in [\![A]\!]$, $y \in [\![\neg A]\!]$ and $y \prec_{\Psi * A} x$. From $y \prec_{\Psi * A} x$, it follows by Success that $x \notin \min(\preccurlyeq_\Psi, [\![A]\!])$ and hence by $x \in [\![A]\!]$ that $x \notin \min(\preccurlyeq_\Psi, [\![A \vee C]\!])$. By $(\beta 2^*_{\preccurlyeq})$, we then recover $y \prec_{\Psi * A \vee C} x$, as required. $\qquad\square$

Proposition 7. *None of* $(\gamma 1^*_{\preccurlyeq})$ *to* $(\gamma 4^*_{\preccurlyeq})$ *follows from AGM,* $(C1^*_{\preccurlyeq})$*,* $(C2^*_{\preccurlyeq})$ *and* $(P^*_{\preccurlyeq})$ *alone.*

Proof:

(a) **Regarding** $(\gamma 1^*_{\preccurlyeq})$ **and** $(\gamma 2^*_{\preccurlyeq})$**:** Consider the countermodel below. It is easily verified that $(C1^*_{\preccurlyeq})$, $(C2^*_{\preccurlyeq})$ and $(P^*_{\preccurlyeq})$ are all satisfied. However, both $(\gamma 1^*_{\preccurlyeq})$ and $(\gamma 2^*_{\preccurlyeq})$ are violated, since $[\![\neg A \wedge \neg C]\!] \prec_{\Psi * A} [\![A \wedge \neg C]\!]$ but $[\![A \wedge \neg C]\!] \prec_{\Psi * A \vee C} [\![\neg A \wedge \neg C]\!]$.
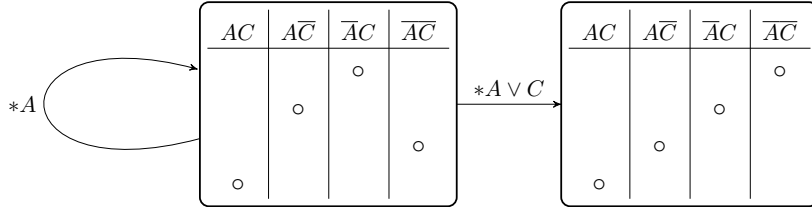
(b) **Regarding** $(\gamma 3^*_{\preccurlyeq})$ **and** $(\gamma 4^*_{\preccurlyeq})$**:** Consider the countermodel below. It is easily verified that $(C1^*_{\preccurlyeq})$, $(C2^*_{\preccurlyeq})$ and $(P^*_{\preccurlyeq})$ are all satisfied. However, both $(\gamma 3^*_{\preccurlyeq})$ and $(\gamma 4^*_{\preccurlyeq})$ are violated, since, although $[\![\neg A \wedge C]\!] \notin \min(\preccurlyeq_\Psi, [\![C]\!])$, we have $[\![\neg A \wedge \neg C]\!] \prec_{\Psi * A \vee C} [\![\neg A \wedge C]\!]$ but $[\![\neg A \wedge C]\!] \prec_{\Psi * C} [\![\neg A \wedge \neg C]\!]$.

$\qquad\square$

$*A \vee C$ →

AC	$A\overline{C}$	$\overline{A}C$	$\overline{A}\,\overline{C}$
	$\circ$		
		$\circ$	
			$\circ$
$\circ$			

$*C$ →

AC	$A\overline{C}$	$\overline{A}C$	$\overline{A}\,\overline{C}$
	$\circ$		
			$\circ$
		$\circ$	
$\circ$			
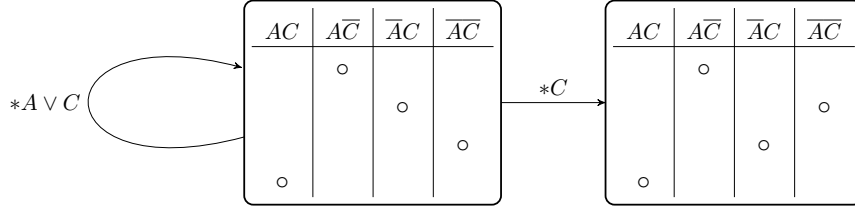
Proposition 8. $(\alpha1^*_{\preccurlyeq})$, $(\alpha2^*_{\preccurlyeq})$ and $(\alpha3^*_{\preccurlyeq})$ are satisfied by POI revision operators.

Proof: We first note that, for POI operators, $y \prec_{\Psi*A} x$ iff

(1) $y \in \min(\preccurlyeq_\Psi, [\![A]\!])$ and $x \notin \min(\preccurlyeq_\Psi, [\![A]\!])$, or

(2) $x, y \notin \min(\preccurlyeq_\Psi, [\![A]\!])$ and $r_A(y) <_\Psi r_A(x)$

We also have $z \prec_{\Psi*C} x$ iff either

(3) $z \in \min(\preccurlyeq_\Psi, [\![C]\!])$ and $x \notin \min(\preccurlyeq_\Psi, [\![C]\!])$, or

(4) $z, x \notin \min(\preccurlyeq_\Psi, [\![C]\!])$ and $r_C(z) <_\Psi r_C(x)$

as well as $y \preccurlyeq_{\Psi*A} x$ iff

(5) $y \in \min(\preccurlyeq_\Psi, [\![A]\!])$, or

(6) $x, y \notin \min(\preccurlyeq_\Psi, [\![A]\!])$ and $r_A(y) \leq_\Psi r_A(x)$

and finally $z \preccurlyeq_{\Psi*C} x$ iff either

(7) $z \in \min(\preccurlyeq_\Psi, [\![C]\!])$, or

(8) $z, x \notin \min(\preccurlyeq_\Psi, [\![C]\!])$ and $r_C(z) \leq_\Psi r_C(x)$

With this in hand, we prove the soundness of each principle in turn:

(a) **Regarding** $(\alpha1^*_{\preccurlyeq})$**:** Assume $x \notin \min(\preccurlyeq_\Psi, [\![C]\!])$, $x \in [\![A]\!]$, $y \in [\![\neg A]\!]$, $z \preccurlyeq_\Psi y$ and $y \preccurlyeq_{\Psi*A} x$. Since $y \in [\![\neg A]\!]$, we have $y \notin \min(\preccurlyeq_\Psi, [\![A]\!])$, placing us in case (6). So $r_A(y) \leq_\Psi r_A(x)$. Since $x \in [\![A]\!]$, we have

$r_A(x) = x^+$ and, since $y \in [\![\neg A]\!]$, it follows that $r_A(y) = y^-$. So $y^- \leq_\Psi x^+$. Furthermore, since $z \preccurlyeq_\Psi y$, we have $z^- \leq_\Psi y^-$. Hence $z^- \leq_\Psi x^+$. Since $z^+ \leq_\Psi z^-$ and $x^+ \leq_\Psi x^-$, it then follows that $r_C(z) \leq_\Psi r_C(x)$. Since we have also assumed $x \notin \min(\preccurlyeq_\Psi, [\![C]\!])$, if $z \in \min(\preccurlyeq_\Psi, [\![C]\!])$, we are in case (7) and if $z \notin \min(\preccurlyeq_\Psi, [\![C]\!])$, we are in case (8). Either way, we have $z \preccurlyeq_{\Psi*C} x$, as required.

(b) **Regarding** $(\alpha 2^*_\preccurlyeq)$**:** Assume $x \notin \min(\preccurlyeq_\Psi, [\![C]\!])$, $x \in [\![A]\!]$, $y \in [\![\neg A]\!]$, $z \preccurlyeq_\Psi y$ and $y \prec_{\Psi*A} x$. Since $y \in [\![\neg A]\!]$, we have $y \notin \min(\preccurlyeq_\Psi, [\![A]\!])$, placing us in case (2). So $r_A(y) <_\Psi r_A(x)$. Since $x \in [\![A]\!]$, we have $r_A(x) = x^+$ and, since $y \in [\![\neg A]\!]$, it follows that $r_A(y) = y^-$. So $y^- <_\Psi x^+$. Furthermore, since $z \preccurlyeq_\Psi y$, we have $z^- \leq_\Psi y^-$. Hence $z^- <_\Psi x^+$. Since $z^+ \leq_\Psi z^-$ and $x^+ \leq_\Psi x^-$, it then follows that $r_C(z) <_\Psi r_C(x)$. Since we have also assumed $x \notin \min(\preccurlyeq_\Psi, [\![C]\!])$, if $z \in \min(\preccurlyeq_\Psi, [\![C]\!])$, we are in case (3) and if $z \notin \min(\preccurlyeq_\Psi, [\![C]\!])$, we are in case (4). Either way, we have $z \prec_{\Psi*C} x$, as required.

(c) **Regarding** $(\alpha 3^*_\preccurlyeq)$**:** Assume $x \notin \min(\preccurlyeq_\Psi, [\![C]\!])$, $x \in [\![A]\!]$, $y \in [\![\neg A]\!]$, $z \prec_\Psi y$ and $y \preccurlyeq_{\Psi*A} x$. Since $y \in [\![\neg A]\!]$, we have $y \notin \min(\preccurlyeq_\Psi, [\![A]\!])$, placing us in case (6). So $r_A(y) \leq_\Psi r_A(x)$. Since $x \in [\![A]\!]$, we have $r_A(x) = x^+$ and, since $y \in [\![\neg A]\!]$, it follows that $r_A(y) = y^-$. So $y^- \leq_\Psi x^+$. Furthermore, since $z \prec_\Psi y$, we have $z^- <_\Psi y^-$. Hence $z^- <_\Psi x^+$. Since $z^+ \leq_\Psi z^-$ and $x^+ \leq_\Psi x^-$, it then follows that $r_C(z) <_\Psi r_C(x)$. Since we have also assumed $x \notin \min(\preccurlyeq_\Psi, [\![C]\!])$, if $z \in \min(\preccurlyeq_\Psi, [\![C]\!])$, we are in case (3) and if $z \notin \min(\preccurlyeq_\Psi, [\![C]\!])$, we are in case (4). Either way, we have $z \prec_{\Psi*C} x$, as required. $\qquad \square$

Theorem 2. $*$ *is a POI revision operator iff it satisfies AGM,* $(\mathrm{Eq}^*_\preccurlyeq)$, $(\mathrm{C1}^*_\preccurlyeq)$, $(\mathrm{C2}^*_\preccurlyeq)$, $(\mathrm{P}^*_\preccurlyeq)$, $(\alpha 1^*_\preccurlyeq)$, $(\alpha 2^*_\preccurlyeq)$, *and* $(\alpha 3^*_\preccurlyeq)$.

Proof: In conjunction with the results of Booth & Meyer regarding non-prioritised POI revision operators, Propositions 2 and 8 establish the left-to-

right direction of the claim. We simply need to establish the other direction.

Recall that, by Definition 7, we need to show that, if $*$ satisfies the relevant semantic properties, then there exists a non-prioritised POI revision operator $\circ$ such that $\mathbb{N}(*, \circ)$.

The construction works as follows: From each $*$ we can construct $\circ$ by setting, for all $x, y \in W$:

$$x \preccurlyeq_{\Psi \circ A} y \text{ iff } x \preccurlyeq_{\Psi * A \vee \neg(x \vee y)} y$$

Note that given $(\text{Eq}^*_{\preccurlyeq})$, $(\text{C1}^*_{\preccurlyeq})$ and $(\text{C2}^*_{\preccurlyeq})$ this is equivalent to:

$$x \preccurlyeq_{\Psi \circ A} y \text{ iff } \begin{cases} x \preccurlyeq_{\Psi} y & \text{if } x \sim^A y \\\\ x \preccurlyeq_{\Psi * \neg y} y & \text{if } x \lhd^A y \\\\ x \preccurlyeq_{\Psi * \neg x} y & \text{if } y \lhd^A x \end{cases}$$

where (i) $x \unlhd^A y$ iff $x \in [\![A]\!]$ or $y \in [\![\neg A]\!]$, (ii) $x \sim^A y$ when $x \unlhd^A y$ and $y \unlhd^A x$, and (iii) $x \lhd^A y$ when $x \unlhd^A y$ but not $y \unlhd^A x$.

We will establish the result by proving two main lemmas: first, we will show that $\circ$ is a non-prioritised POI revision operator (Lemma 1) and then we will show that $\mathbb{N}(*, \circ)$ (Lemma 3).

Lemma 1. $\circ$ *is a non-prioritised POI revision operator*

We show that $\circ$ satisfies each of $(\text{C1}^\circ_{\preccurlyeq})$, $(\text{C2}^\circ_{\preccurlyeq})$, $(\text{P}^\circ_{\preccurlyeq})$, $(\text{s}\beta 1^\circ_{\preccurlyeq})$ and $(\text{s}\beta 2^\circ_{\preccurlyeq})$, as well as the requirement that $\preccurlyeq_{\Psi \circ A}$ is a TPO over W. Before doing so, however, we first establish the following useful auxiliary lemma:

Lemma 2. *Let x, y, z be distinct worlds such that $y \preccurlyeq_{\Psi} z$. Then the following are equivalent:*

(i) *If $x \preccurlyeq_{\Psi * \neg y} y$, then $x \preccurlyeq_{\Psi * \neg z} z$*

(ii) *If $x \preccurlyeq_{\Psi * x \vee z} y$, then $x \preccurlyeq_{\Psi * x \vee y} z$*

The proof of Lemma 2 is as follows:

(a) **From (i) to (ii):** Suppose $z \prec_{\Psi * x \vee z} x$. Then by $(\gamma 2_{\preccurlyeq}^{*})$ and $(\mathrm{Eq}_{\preccurlyeq}^{*})$, we have $z \prec_{\Psi * \neg z} x$. Hence $y \prec_{\Psi * \neg y} x$, by (i). From $z \prec_{\Psi * \neg z} x$ we also know that $z \prec_{\Psi} x$ by $(\mathrm{C3}_{\preccurlyeq}^{*})$. Hence $x \notin \min(\preccurlyeq_{\Psi}, \llbracket x \vee z \rrbracket)$ and so, from $y \prec_{\Psi * \neg y} x$, we can conclude $y \prec_{\Psi * x \vee z} x$ by $(\beta 2_{\preccurlyeq}^{*})$.

(b) **From (ii) to (i):** Suppose $z \preccurlyeq_{\Psi * \neg z} x$. Then $z \preccurlyeq_{\Psi} x$ by $(\mathrm{C3}_{\preccurlyeq}^{*})$, so, since we can also assume $y \preccurlyeq_{\Psi} z$, $x \preccurlyeq_{\Psi} z$ and therefore $x \notin \min(\preccurlyeq_{\Psi}, \llbracket x \vee y \rrbracket)$. Then, from this and $z \prec_{\Psi * \neg z} x$, we obtain $z \prec_{\Psi * x \vee y} x$ by postulate $(\beta 2_{\preccurlyeq}^{*})$. Hence from (ii), $y \prec_{\Psi * x \vee z} x$ and then $y \prec_{\Psi * \neg y} x$ by $(\gamma 2_{\preccurlyeq}^{*})$ and $(\mathrm{Eq}_{\preccurlyeq}^{*})$.

We now return to the proof of Lemma 1.

(a) **Regarding $\preccurlyeq_{\Psi \circ A}$'s being a TPO over W:** We have $x \preccurlyeq_{\Psi \circ A} y \Leftrightarrow x \preccurlyeq_{\Psi * A \vee \neg(x \vee y)} y$. So completeness of $\preccurlyeq_{\Psi \circ A}$ follows from completeness of $\preccurlyeq_{\Psi * A \vee \neg(x \vee y)}$ and $(\mathrm{Eq}_{\preccurlyeq}^{*})$. To show that $\preccurlyeq_{\Psi \circ A}$ is transitive (i.e. that, if $x \preccurlyeq_{\Psi \circ A} y$ and $y \preccurlyeq_{\Psi \circ A} z$, then $x \preccurlyeq_{\Psi \circ A} z$), we go through the 8 cases according to whether each of x, y, and z is in $\llbracket A \rrbracket$ or not:

   (i) $x, y, z \in \llbracket A \rrbracket$ **or** $x, y, z \in \llbracket \neg A \rrbracket$: Follows from transitivity for $*$.

   (ii) $x, y \in \llbracket A \rrbracket$, $z \in \llbracket \neg A \rrbracket$: Then we must show that, if $x \preccurlyeq_{\Psi} y$ and $y \preccurlyeq_{\Psi * \neg z} z$, then $x \preccurlyeq_{\Psi * \neg z} z$. Since $x, y \in \llbracket A \rrbracket$ and $z \in \llbracket \neg A \rrbracket$, we know that $x \neq z$ and $y \neq z$. Then from $x \preccurlyeq_{\Psi} y$ and $(\mathrm{C1}_{\preccurlyeq}^{*})$, we obtain $x \preccurlyeq_{\Psi * \neg z} y$. From the latter and $y \preccurlyeq_{\Psi * \neg z} z$, we then obtain $x \preccurlyeq_{\Psi * \neg z} z$ by transitivity for $*$.

   (iii) $x \in \llbracket A \rrbracket$, $y \in \llbracket \neg A \rrbracket$, $z \in \llbracket A \rrbracket$: Then we must show that, if $x \preccurlyeq_{\Psi * \neg y} y$ and $y \preccurlyeq_{\Psi * \neg y} z$, then $x \preccurlyeq_{\Psi} z$. By transitivity for $*$, it follows, from $x \preccurlyeq_{\Psi * \neg y} y$ and $y \preccurlyeq_{\Psi * \neg y} z$, that $x \preccurlyeq_{\Psi * \neg y} z$. From $x, z \in \llbracket A \rrbracket$ and $y \in \llbracket \neg A \rrbracket$, we know $x \neq y$ and $z \neq y$. So from $x \preccurlyeq_{\Psi * \neg y} z$ and $(\mathrm{C1}_{\preccurlyeq}^{*})$, we obtain $x \preccurlyeq_{\Psi} z$.

   (iv) $x \in \llbracket A \rrbracket$, $y, z \in \llbracket \neg A \rrbracket$: Then we must show that, if $x \preccurlyeq_{\Psi * \neg y} y$ and $y \preccurlyeq_{\Psi} z$, then $x \preccurlyeq_{\Psi * \neg z} z$. If $z = y$, then $x \preccurlyeq_{\Psi * \neg z} z$ follows

immediately from $x \preccurlyeq_{\Psi * \neg y} y$. So we may assume $z \neq y$. By Lemma 2, what we must establish is then equivalent to: if $x \preccurlyeq_{\Psi * x \vee z} y$ and $y \preccurlyeq_\Psi z$, then $x \preccurlyeq_{\Psi * x \vee y} z$. Or contraposing: if $z \prec_{\Psi * x \vee y} x$ and $y \preccurlyeq_\Psi z$, then $y \prec_{\Psi * x \vee z} x$. So assume $z \prec_{\Psi * x \vee y} x$ and $y \preccurlyeq_\Psi z$. Now, if $x \preccurlyeq_\Psi z$, then $x \preccurlyeq_{\Psi * x \vee y} z$ by (C3$^*_{\preccurlyeq}$). So assume $z \prec_\Psi x$. We therefore have: $x \in [\![x \vee y]\!]$, $z \notin [\![x \vee y]\!]$, $z \prec_{\Psi * x \vee y} x$, $y \preccurlyeq_\Psi z$ and $x \notin \min(\preccurlyeq_\Psi, [\![x \vee z]\!])$. From this, by ($\alpha 2^*_{\preccurlyeq}$), we can then infer that $y \prec_{\Psi * x \vee z} x$, as required.

(v) $x \in [\![\neg A]\!]$, $y, z \in [\![A]\!]$: Then we must show that, if $x \preccurlyeq_{\Psi * \neg x} y$ and $y \preccurlyeq_\Psi z$, then $x \preccurlyeq_{\Psi * \neg x} z$. Since $x \in [\![\neg A]\!]$ and $y, z \in [\![A]\!]$, we know that $x \neq y$ and $x \neq z$. Hence from $y \preccurlyeq_\Psi z$, we know $y \preccurlyeq_{\Psi * \neg x} z$. The desired implication then follows from transitivity for $*$.

(vi) $x \in [\![\neg A]\!]$, $y \in [\![A]\!]$, $z \in [\![\neg A]\!]$: Then we must show that, if $x \preccurlyeq_{\Psi * \neg x} y$ and $y \preccurlyeq_{\Psi * \neg z} z$, then $x \preccurlyeq_\Psi z$, or, equivalently, that, if $x \preccurlyeq_{\Psi * \neg x} y$ and $z \prec_\Psi x$, then $z \prec_{\Psi * \neg z} y$. So suppose $x \preccurlyeq_{\Psi * \neg x} y$ and $z \prec_\Psi x$. If $y \preccurlyeq_\Psi x$, then by (P$^*_{\preccurlyeq}$) we would have $y \prec_{\Psi * \neg x} x$: contradiction. Hence we may assume $x \prec_\Psi y$. From this and $z \prec_\Psi x$ we have, by transitivity, $z \prec_\Psi y$ and hence $y \notin \min(\preccurlyeq_\Psi, [\![y \vee z]\!])$. From this and $x \preccurlyeq_{\Psi * \neg x} y$, using postulate ($\beta 1^*_{\preccurlyeq}$), we can deduce $x \preccurlyeq_{\Psi * y \vee z} y$. We therefore have: $y \in [\![y \vee z]\!]$, $x \notin [\![y \vee z]\!]$, $x \preccurlyeq_{\Psi * y \vee z} y$, $z \prec_\Psi x$ and $y \notin \min(\preccurlyeq_\Psi, [\![x \vee y]\!])$. From this, by ($\alpha 3^*_{\preccurlyeq}$), we can then infer that $z \prec_{\Psi * x \vee y} x$, and so $z \prec_{\Psi * \neg z} y$, by ($\gamma 2^*_{\preccurlyeq}$), as required.

(vii) $x, y \in [\![\neg A]\!]$, $z \in [\![A]\!]$: Then we must show that, if $x \preccurlyeq_\Psi y$ and $y \preccurlyeq_{\Psi * \neg y} z$, then $x \preccurlyeq_{\Psi * \neg x} z$. If $x = y$, then this holds immediately, so we may assume $x \neq y$. Now suppose $x \preccurlyeq_\Psi y$ and $y \preccurlyeq_{\Psi * \neg y} z$. If $z \preccurlyeq_\Psi y$, then $z \prec_{\Psi * \neg y} y$ by (P$^*_{\preccurlyeq}$): contradiction. So we may assume $y \prec_\Psi z$. From this and

$x \preccurlyeq_\Psi y$, we know, by transitivity, that $x \prec_\Psi z$, so $z \notin \min(\preccurlyeq_\Psi$
$, [\![x \vee z]\!])$. It then follows that $y \preccurlyeq_{\Psi * x \vee z} z$ by postulate $(\beta 1^*_{\preccurlyeq})$.
We therefore have: $z \in [\![x \vee z]\!]$, $y \notin [\![x \vee z]\!]$, $y \preccurlyeq_{\Psi * x \vee z} z$,
$x \prec_\Psi y$ and $z \notin \min(\preccurlyeq_\Psi, [\![y \vee z]\!])$. From this, by $(\alpha 1^*_{\preccurlyeq})$, we
can then infer that $x \prec_{\Psi * y \vee z} z$, and so $x \preccurlyeq_{\Psi * \neg x} z$, by $(\gamma 1^*_{\preccurlyeq})$,
as required.

(b) **Regarding** $(\text{C1}^\circ_{\preccurlyeq})$ **&** $(\text{C2}^\circ_{\preccurlyeq})$**:** we have already noted towards the beginning of the proof that $x \preccurlyeq_{\Psi \circ A} y$ iff $x \preccurlyeq_\Psi y$, whenever $x \sim^A y$.

(c) **Regarding** $(\text{P}^\circ_{\preccurlyeq})$**:** Suppose $x \vartriangleleft^A y$ and $x \preccurlyeq_\Psi y$. We must show that $x \preccurlyeq_{\Psi \circ A} y$ and $y \not\preccurlyeq_{\Psi \circ A} x$. For this, it suffices to show that $x \preccurlyeq_{\Psi * \neg y} y$ and $y \not\preccurlyeq_{\Psi * \neg y} x$, i.e. that $x \prec_{\Psi * \neg y} y$. This follows from $(\text{P}^*_{\preccurlyeq})$.

(d) **Regarding** $(\text{s}\beta 1^\circ_{\preccurlyeq})$ **&** $(\text{s}\beta 2^\circ_{\preccurlyeq})$**:** Proposition 3 of [12] tells us that, if $\circ$ satisfies the previous properties, then $(\text{s}\beta 1^\circ_{\preccurlyeq})$ and $(\text{s}\beta 2^\circ_{\preccurlyeq})$ are jointly equivalent to the following condition:

(IIA^*) If A and B agree on x and y, then $x \preccurlyeq_{\Psi \circ A} y$ iff $x \preccurlyeq_{\Psi \circ B} y$

where, given $A, B \in L$ and $x, y \in W$, A and B are said to *agree* on x and y iff either (i) $x \vartriangleleft^A y$ and $x \vartriangleleft^B y$, (ii) $x \sim^A y$ and $x \sim^B y$ or (iii) $y \vartriangleleft^A x$ and $y \vartriangleleft^B x$. Hence it suffices to show that $\circ$ satisfies (IIA^*). But this is immediate from our characterisation of $\circ$ towards the beginning of this proof:

$$ x \preccurlyeq_{\Psi \circ A} y \text{ iff } \begin{cases} x \preccurlyeq_\Psi y & \text{if } x \sim^A y \\ \\ x \preccurlyeq_{\Psi * \neg y} y & \text{if } x \vartriangleleft^A y \\ \\ x \preccurlyeq_{\Psi * \neg x} y & \text{if } y \vartriangleleft^A x \end{cases} $$

We now prove our second main lemma:

Lemma 3. $\mathbb{N}(*, \circ)$

We require:

$x \preccurlyeq_{\Psi * A} y$ iff

(i) $x \in \min(\preccurlyeq_{\Psi \circ A}, \llbracket A \rrbracket)$, or

(ii) $x, y \notin \min(\preccurlyeq_{\Psi \circ A}, \llbracket A \rrbracket)$ and $x \preccurlyeq_{\Psi * A \vee \neg(x \vee y)} y$

We can however replace this with

$x \preccurlyeq_{\Psi * A} y$ iff

(i) $x \in \min(\preccurlyeq_{\Psi}, \llbracket A \rrbracket)$, or

(ii) $x, y \notin \min(\preccurlyeq_{\Psi}, \llbracket A \rrbracket)$ and $x \preccurlyeq_{\Psi * A \vee \neg(x \vee y)} y$

since $\circ$ satisfies (C1$^{\circ}_{\preccurlyeq}$).

(a) **Regarding the left-to-right direction:** Suppose that $x \preccurlyeq_{\Psi * A} y$ and $x \notin \min(\preccurlyeq_{\Psi}, \llbracket A \rrbracket)$. If $y \in \min(\preccurlyeq_{\Psi}, \llbracket A \rrbracket)$, then $y \prec_{\Psi * A} x$, by Success: contradiction. Hence $y \notin \min(\preccurlyeq_{\Psi}, \llbracket A \rrbracket)$. It remains to be shown that $x \preccurlyeq_{\Psi * A \vee \neg(x \vee y)} y$. If $x \sim^A y$, then the conclusion follows by (C1$^{*}_{\preccurlyeq}$)–(C2$^{*}_{\preccurlyeq}$). If $y \lhd^A x$, then the conclusion follows from $x \preccurlyeq_{\Psi * A} y$ and ($\gamma 1^{*}_{\preccurlyeq}$). Finally, if $x \lhd^A y$, then $x \in \llbracket A \vee \neg(x \vee y) \rrbracket$ and $y \in \llbracket \neg(A \vee \neg(x \vee y)) \rrbracket$. Together with $x \notin \min(\preccurlyeq_{\Psi}, \llbracket A \rrbracket)$ and $x \preccurlyeq_{\Psi * A} y$, the desired conclusion then follows by postulate ($\beta 2^{*}_{\preccurlyeq}$).

(b) **Regarding the right-to-left direction:** If $x \in \min(\preccurlyeq_{\Psi}, \llbracket A \rrbracket)$, then $x \preccurlyeq_{\Psi * A} y$ by Success. So suppose $x, y \notin \min(\preccurlyeq_{\Psi}, \llbracket A \rrbracket)$ and $x \preccurlyeq_{\Psi * A \vee \neg(x \vee y)} y$. We must show $x \preccurlyeq_{\Psi * A} y$. If $x \sim^A y$, then the conclusion follows by (C1$^{*}_{\preccurlyeq}$)–(C2$^{*}_{\preccurlyeq}$). If $x \lhd^A y$, then the conclusion follows by ($\gamma 2^{*}_{\preccurlyeq}$). Finally, if $y \lhd^A x$, then the conclusion follows from postulate ($\beta 1^{*}_{\preccurlyeq}$). $\square$

Proposition 12. *If $*$ is a POI operator, then:*

(a) $A \notin [(\Psi * A) * B]$ iff $\min(\lll_{\Psi}, \llbracket B \rrbracket) \subseteq \llbracket \neg A \rrbracket$ and $\neg B \in [\Psi * A]$, and

54

(b) $\neg A \in [(\Psi * A) * B]$ *iff* $\min(\lll_\Psi, [\![B]\!]) \subseteq [\![\neg A]\!]$ *and* $\neg B \in [\Psi * A]$.

Proof: We simply provide the proof of (a), since the proof of (b) is analogous. The derivation is similar to the one provided for Proposition 8, Part (i), of Booth & Meyer [12]. We first note that $A \notin [(\Psi * A) * B]$ is given semantically by $\min(\preccurlyeq_{\Psi*A}, [\![B]\!]) \nsubseteq [\![A]\!]$ and that $\neg B \in [\Psi * A]$ is given by $\min(\preccurlyeq_\Psi, [\![A]\!]) \subseteq [\![\neg B]\!]$.

(i) **Regarding the left-to-right direction:** Assume the truth of the antecedent, i.e. $\min(\preccurlyeq_{\Psi*A}, [\![B]\!]) \nsubseteq [\![A]\!]$, so that there exists $y \in \min(\preccurlyeq_{\Psi*A}, [\![B]\!]) \cap [\![\neg A]\!]$. Suppose for reductio that the consequent is false, so that either

   (a) $\min(\lll_\Psi, [\![B]\!]) \nsubseteq [\![\neg A]\!]$ or

   (b) $\min(\preccurlyeq_\Psi, [\![A]\!]) \nsubseteq [\![\neg B]\!]$.

   Assume (a), so that there exists $x \in \min(\lll_\Psi, [\![B]\!]) \cap [\![A]\!]$. By $y \in \min(\preccurlyeq_{\Psi*A}, [\![B]\!])$ and $x \in [\![B]\!]$, we have $y \preccurlyeq_{\Psi*A} x$, which, by the definition of POI assignments, amounts to:

   (1) $y \in \min(\preccurlyeq_\Psi, [\![A]\!])$, or

   (2) $x, y \notin \min(\preccurlyeq_\Psi, [\![A]\!])$ and $r_A(y) \leq_\Psi r_A(x)$

   Since $y \in [\![\neg A]\!]$, (1) is ruled out, placing us in case (2). Since $x \in [\![A]\!]$ and $y \in [\![\neg A]\!]$, $r_A(y) \leq_\Psi r_A(x)$ amounts to $y^- \leq_\Psi x^+$, and hence $y \lll_\Psi x$, contradicting $x \in \min(\lll_\Psi, [\![B]\!])$.

   Assume (b), so that there exists $z \in \min(\preccurlyeq_\Psi, [\![A]\!]) \cap [\![B]\!]$. Then $\min(\preccurlyeq_{\Psi*A}, [\![B]\!]) = \min(\preccurlyeq_\Psi, [\![A]\!]) \cap [\![B]\!]$. But, since $\min(\preccurlyeq_\Psi, [\![A]\!]) \cap [\![B]\!] \subseteq [\![A]\!]$, we then have $\min(\preccurlyeq_{\Psi*A}, [\![B]\!]) \subseteq [\![A]\!]$, directly contradicting out initial assumption.

(ii) **Regarding the right-to-left direction:** Assume the antecedent, so that:

   (a) $\min(\lll_\Psi, [\![B]\!]) \subseteq [\![\neg A]\!]$ and

(b) $\min(\preccurlyeq_\Psi, [\![A]\!]) \subseteq [\![\neg B]\!]$.

We will show that there exists y in $\min(\lll_\Psi, [\![B]\!])$, and hence in $[\![\neg A]\!]$, that is also in $\min(\preccurlyeq_{\Psi*A}, [\![B]\!])$, which will suffice to establish the consequent. Assume then for reductio that for all $y \in \min(\lll_\Psi, [\![B]\!])$, we have $y \notin \min(\preccurlyeq_{\Psi*A}, [\![B]\!])$. So there exists $x \in [\![B]\!]$ such that, for all $y \in \min(\lll_\Psi, [\![B]\!])$, $x \prec_{\Psi*A} y$. Now, by the definition of POI assignments, $x \prec_{\Psi*A} y$ holds iff:

(1) $x \in \min(\preccurlyeq_\Psi, [\![A]\!])$ and $y \notin \min(\preccurlyeq_\Psi, [\![A]\!])$, or

(2) $x, y \notin \min(\preccurlyeq_\Psi, [\![A]\!])$ and $r_A(x) <_\Psi r_A(y)$

Given (b) and $x \in [\![B]\!]$, it follows that $x \notin \min(\preccurlyeq_\Psi, [\![A]\!])$ and so (1) fails. This places us in case (2). Since $y \in \min(\lll, [\![B]\!])$, by (a), we have $y \in [\![\neg A]\!]$ and so $r_A(x) <_\Psi r_A(y)$ gives us $r_A(x) <_\Psi y^-$. From $r_A(x) <_\Psi r_A(y)$, we also have $x \neq y$, and so $x \notin \min(\lll_\Psi, [\![B]\!])$. So, for some $y \in \min(\lll_\Psi, [\![B]\!])$, we must have $y \lll_\Psi x$, that is, $y^- \leq_\Psi x^+$. But this contradicts $r_A(x) <_\Psi y^-$.

Proposition 10. $*$ *is a POI revision operator iff it satisfies AGM,* (Eq^*), $(\mathrm{C1}^*)$, $(\mathrm{C2}^*)$, $(\beta 1^*)$, $(\beta 2^*)$, $(\Omega 1^*)$–$(\Omega 3^*)$.

Proof:

Given Theorem 2, it will suffice to establish that the following are equivalent: (a) $(\mathrm{Eq}^*_\preccurlyeq)$, $(\mathrm{C1}^*_\preccurlyeq)$, $(\mathrm{C2}^*_\preccurlyeq)$, $(\mathrm{P}^*_\preccurlyeq)$, $(\alpha 1^*_\preccurlyeq)$–$(\alpha 3^*_\preccurlyeq)$, and (b) (Eq^*), $(\mathrm{C1}^*)$, $(\mathrm{C2}^*)$, $(\beta 1^*)$, $(\beta 2^*)$, $(\Omega 1^*)$–$(\Omega 3^*)$.

The equivalence of $(\mathrm{C1}^*)$ and $(\mathrm{C2}^*)$ to $(\mathrm{C1}^*_\preccurlyeq)$ and $(\mathrm{C2}^*_\preccurlyeq)$ is well known. So we first show that, given $(\mathrm{P}^*_\preccurlyeq)$, $(\alpha i^*_\preccurlyeq)$ entails (Ωi^*), for $1 \leq i \leq 3$.

(i) **Regarding** $(\Omega 1^*)$**:** From $A \notin [(\Psi * A) * B]$ we know there exists $y \in [\![\neg A]\!] \cap \min(\preccurlyeq_{\Psi*A}, [\![B]\!])$. From $\neg A \notin [\Psi * A \vee B]$ there exists $z \in [\![A]\!] \cap \min(\preccurlyeq_\Psi, [\![A \vee B]\!])$. From the minimality of z we know $z \preccurlyeq_\Psi y$. If it were the case that $z \in [\![B]\!]$ then $y \preccurlyeq_{\Psi*A} z$ by the minimality of y

56

and so we must have $y \prec_\Psi z$ by $(\mathrm{P}^*_{\preccurlyeq})$–contradicting the minimality of z. Hence $z \in [\![\neg B]\!]$. Now assume for contradiction $B \in [(\Psi * B) * A]$ and let $x \in \min(\preccurlyeq_{\Psi * B}, [\![A]\!])$. Then $x \in [\![B]\!]$ and, since $z \in [\![A \wedge \neg B]\!]$, $x \prec_{\Psi * B} z$. Since $y \in \min(\preccurlyeq_{\Psi * A}, [\![B]\!])$ we have $y \preccurlyeq_{\Psi * A} x$ and so also $y \prec_\Psi x$ by $(\mathrm{P}^*_{\preccurlyeq})$ which gives $x \notin \min(\preccurlyeq_\Psi, [\![B]\!])$. We have now established $x \in [\![A]\!]$, $y \in [\![\neg A]\!]$, $y \preccurlyeq_{\Psi * A} x$, $z \preccurlyeq_\Psi y$ and $x \notin \min(\preccurlyeq_\Psi, [\![B]\!])$. Hence we may deduce, by $(\alpha 1^*_{\preccurlyeq})$, that $z \preccurlyeq_{\Psi * B} x$, contradicting what we already established. Hence $B \notin [(\Psi * B) * A]$.

(ii) **Regarding** $(\Omega 2^*)$**:** Assume for contradiction $\neg B \notin [(\Psi * B) * A]$. Then there exists $x \in [\![B]\!] \cap \min(\preccurlyeq_{\Psi * B}, [\![A]\!])$. From $\neg A \in [(\Psi * A) * B]$ we know $x \notin \min(\preccurlyeq_{\Psi * A}, [\![B]\!])$. Let $y \in \min(\preccurlyeq_{\Psi * A}, [\![B]\!])$. Then $y \prec_{\Psi * A} x$ and $y \in [\![\neg A]\!]$. From $y \prec_{\Psi * A} x$ we also know $y \prec_\Psi x$ by $(\mathrm{C}4^*_{\preccurlyeq})$ (which follows from $(\alpha 2^*_{\preccurlyeq})$), so $x \notin \min(\preccurlyeq_\Psi, [\![B]\!])$. From $\neg A \notin [\Psi * A \vee B]$ there exists $z \in [\![A]\!] \cap \min(\preccurlyeq_\Psi, [\![A \vee B]\!])$. Since $y \in [\![B]\!]$ we have $z \preccurlyeq_\Psi y$. So we have established $x \in [\![A]\!]$, $y \in [\![\neg A]\!]$, $y \prec_{\Psi * A} x$, $z \preccurlyeq_\Psi y$ and $x \notin \min(\preccurlyeq_\Psi, [\![B]\!])$. We can then apply $(\alpha 2^*_{\preccurlyeq})$ to deduce $z \prec_{\Psi * B} x$, contradicting the minimality of x. Hence $\neg B \in [(\Psi * B) * A]$ as required.

(iii) **Regarding** $(\Omega 3^*)$**:** From $A \notin [(\Psi * A) * B]$ there exists $y \in [\![\neg A]\!] \cap \min(\preccurlyeq_{\Psi * A}, [\![B]\!])$. Assume for contradiction $\neg B \notin [(\Psi * B) * A]$. Then there exists $x \in [\![B]\!] \cap \min(\preccurlyeq_{\Psi * B}, [\![A]\!])$. By the minimality of y we know $y \preccurlyeq_{\Psi * A} x$. Since $y \in [\![\neg A]\!]$ and $x \in [\![A]\!]$ this in turn gives $y \prec_\Psi x$ by $(\mathrm{P}^*_{\preccurlyeq})$, so $x \notin \min(\preccurlyeq_\Psi, [\![B]\!])$. Since $y \in [\![B]\!]$ and $\neg B \in [\Psi * A \vee B]$ there must exist some $z \in [\![A \wedge \neg B]\!]$ such that $z \prec_\Psi y$. So we have established $x \in [\![A]\!]$, $y \in [\![\neg A]\!]$, $y \preccurlyeq_{\Psi * A} x$, $z \prec_\Psi y$ and $x \notin \min(\preccurlyeq_\Psi, [\![B]\!])$. Hence we may apply $(\alpha 3^*_{\preccurlyeq})$ and deduce $z \prec_{\Psi * B} x$, contradicting the minimality of x. Hence $\neg B \in [(\Psi * B) * A]$ as required.

Assuming AGM in the background, we now first show that, given (Eq^*), $(\mathrm{C}1^*)$, $(\mathrm{C}2^*)$, $(\beta 1^*)$ and $(\beta 2^*)$, (Ωi^*) entails $(\alpha i^*_{\preccurlyeq})$, for $1 \le i \le 3$. We then show that $(\Omega 1^*)$ entails $(\mathrm{P}^*_{\preccurlyeq})$.

(a) (i) **Regarding** $(\alpha1^*_{\preccurlyeq})$**:** First note that from the assumptions we already obtain $y \preccurlyeq_{\Psi*C} x$ from $(\beta1^*_{\preccurlyeq})$. If $y = z$ then this clearly gives us the required conclusion, so we may assume $y \neq z$. Now, from $x \in \llbracket A \rrbracket$, $y \in \llbracket \neg A \rrbracket$ and $y \preccurlyeq_{\Psi*A} x$, we know $y \prec_{\Psi} x$, by (P^*). Hence we have established $z \preccurlyeq_{\Psi} y \prec_{\Psi} x$. If $z \in \llbracket C \rrbracket$ or $x \in \llbracket \neg C \rrbracket$, then from $z \prec_{\Psi} x$ we obtain $z \prec_{\Psi*C} x$ from $(C1^*_{\preccurlyeq})$, $(C2^*_{\preccurlyeq})$ or $(C3^*_{\preccurlyeq})$ (which follows from $(P^*_{\preccurlyeq})$) and so we obtain the required conclusion $z \preccurlyeq_{\Psi*C} x$. So assume $z \in \llbracket \neg C \rrbracket$ and $x \in \llbracket C \rrbracket$. If $y \in \llbracket \neg C \rrbracket$, then, from $z \in \llbracket \neg C \rrbracket$ and $z \preccurlyeq_{\Psi} y$, we obtain $z \preccurlyeq_{\Psi*C} y$ by $(C2^*_{\preccurlyeq})$, so the required conclusion follows from this, given $y \preccurlyeq_{\Psi*C} x$ and transitivity. So assume $y \in \llbracket C \rrbracket$. If $z \in \llbracket \neg A \rrbracket$, then, since $y \in \llbracket \neg A \rrbracket$, we obtain $z \preccurlyeq_{\Psi*A} y$, by $(C2^*_{\preccurlyeq})$. So $z \preccurlyeq_{\Psi*A} x$ by transitivity with the assumption $y \preccurlyeq_{\Psi*A} x$. We can then apply $(\beta1^*_{\preccurlyeq})$, using this together with the assumptions $x \in \llbracket A \rrbracket$, $z \in \llbracket \neg A \rrbracket$ and $x \notin \min(\preccurlyeq_{\Psi}, \llbracket C \rrbracket)$, to obtain the desired result that $z \preccurlyeq_{\Psi*C} x$. So assume $z \in \llbracket A \rrbracket$. We have now built up the following assumptions about x, y, z: (i) $x \in \llbracket A \wedge C \rrbracket$, (ii) $y \in \llbracket \neg A \wedge C \rrbracket$, and (iii) $z \in \llbracket A \wedge \neg C \rrbracket$. To show the desired result that $z \preccurlyeq_{\Psi*C} x$ in this final case, it suffices, by $(\gamma1^*_{\preccurlyeq})$ and $(Eq^*_{\preccurlyeq})$, to show $z \preccurlyeq_{\Psi*x\vee y} x$, which is equivalent to $x \vee y \notin \llbracket (\Psi * x \vee y) * x \vee z \rrbracket$ (since we assume $z \neq y$ and we know also $z \neq x$ from $z \prec_{\Psi} x$). To prove this it suffices, by $(\Omega1^*)$ and $(Eq^*_{\preccurlyeq})$, to show $\neg(x \vee z) \notin \llbracket \Psi * x \vee y \vee z \rrbracket$ and $x \vee z \notin \llbracket (\Psi * x \vee z) * x \vee y \rrbracket$. But the former holds since we have already established $z \preccurlyeq_{\Psi} y \prec_{\Psi} x$, while the latter is equivalent to $y \preccurlyeq_{\Psi*x\vee z} x$. This will follow from $y \preccurlyeq_{\Psi*A} x$ and $(\beta1^*_{\preccurlyeq})$, provided we have $x \notin \min(\preccurlyeq_{\Psi}, \llbracket x \vee z \rrbracket)$, i.e. $z \prec_{\Psi} x$. But we have already established that.

(ii) **Regarding** $(\alpha2^*_{\preccurlyeq})$**:** First note that, from the assumptions, we already obtain $y \prec_{\Psi*C} x$ from $(\beta2^*_{\preccurlyeq})$. If $y = z$, then this clearly gives us the required conclusion. So we may assume $y \neq z$. Now,

58

from $x \in [\![A]\!]$, $y \in [\![\neg A]\!]$ and $y \prec_{\Psi*A} x$, we know $y \prec_\Psi x$, by (C4$^*_{\preccurlyeq}$). Hence we have established $z \preccurlyeq_\Psi y \prec_\Psi x$. If $z \in [\![C]\!]$ or $x \in [\![\neg C]\!]$, then from $z \prec_\Psi x$ we obtain $z \prec_{\Psi*C} x$, by (C1$^*_{\preccurlyeq}$), (C2$^*_{\preccurlyeq}$) or (C3$^*_{\preccurlyeq}$), as required. So assume $z \in [\![\neg C]\!]$ and $x \in [\![C]\!]$. If $y \in [\![\neg C]\!]$, then, from $z \in [\![\neg C]\!]$ and $z \preccurlyeq_\Psi y$, we obtain $z \preccurlyeq_{\Psi*C} y$, by (C2$^*_{\preccurlyeq}$). So the required conclusion follows from this with $y \prec_{\Psi*C} x$ and transitivity. So assume $y \in [\![C]\!]$. If $z \in [\![\neg A]\!]$ then, since $y \in [\![\neg A]\!]$, we obtain $z \preccurlyeq_{\Psi*A} y$ by (C2$^*_{\preccurlyeq}$). So $z \prec_{\Psi*A} x$, by transitivity, with the assumption $y \prec_{\Psi*A} x$. We can then apply ($\beta2^*_{\preccurlyeq}$), using this together with the assumptions $x \in [\![A]\!]$, $z \in [\![\neg A]\!]$ and $x \notin \min(\preccurlyeq_\Psi, [\![C]\!])$, to obtain the desired $z \prec_{\Psi*C} x$. So assume $z \in [\![A]\!]$. We now have built up the following assumptions about x, y, z: (i) $x \in [\![A \wedge C]\!]$, (ii) $y \in [\![\neg A \wedge C]\!]$, and (iii) $z \in [\![A \wedge \neg C]\!]$. To show the desired result that $z \prec_{\Psi*C} x$ in this final case, it suffices, by ($\gamma2^*_{\preccurlyeq}$) and (Eq$^*_{\preccurlyeq}$), to show $z \prec_{\Psi*x\vee y} x$, which is equivalent to $\neg(x \vee y) \in [(\Psi * x \vee y) * x \vee z]$ (since we assume $z \neq y$ and we know also $z \neq x$ from $z \prec_\Psi x$). To prove this it suffices, by ($\Omega2^*$) and (Eq$^*_{\preccurlyeq}$), to show $\neg(x \vee z) \notin [\Psi * x \vee y \vee z]$ and $\neg(x \vee z) \in [(\Psi * x \vee z) * x \vee y]$. But the former holds since we already established $z \preccurlyeq_\Psi y \prec_\Psi x$, while the latter is equivalent to $y \prec_{\Psi*x\vee z} x$. This will follow from $y \prec_{\Psi*A} x$ and ($\beta2^*_{\preccurlyeq}$), provided we have $x \notin \min(\preccurlyeq_\Psi, [\![x \vee z]\!])$, i.e. $z \prec_\Psi x$. But we have already established that.

(iii) **Regarding** ($\alpha3^*_{\preccurlyeq}$)**:** From $x \in [\![A]\!]$, $y \in [\![\neg A]\!]$ and $y \preccurlyeq_{\Psi*A} x$, we know $y \preccurlyeq_\Psi x$ by (C3$^*_{\preccurlyeq}$). Hence we have established $z \prec_\Psi y \preccurlyeq_\Psi x$. If $z \in [\![C]\!]$ or $x \in [\![\neg C]\!]$, then, from $z \prec_\Psi x$, we obtain $z \prec_{\Psi*C} x$, by (C1$^*_{\preccurlyeq}$), (C2$^*_{\preccurlyeq}$) or (C3$^*_{\preccurlyeq}$), as required. So assume $z \in [\![\neg C]\!]$ and $x \in [\![C]\!]$. From the assumptions, we already know $y \preccurlyeq_{\Psi*C} x$, by ($\beta1^*_{\preccurlyeq}$). If $y \in [\![\neg C]\!]$, then, from $z \in [\![\neg C]\!]$ and $z \prec_\Psi y$, we obtain $z \prec_{\Psi*C} y$, by (C2$^*_{\preccurlyeq}$). So the required conclusion follows from this,

given $y \preccurlyeq_{\Psi * C} x$ and transitivity. So assume $y \in \llbracket C \rrbracket$. If $z \in \llbracket \neg A \rrbracket$, then, since $y \in \llbracket \neg A \rrbracket$, we obtain $z \prec_{\Psi * A} y$, by (C2$_{\preccurlyeq}^*$). So $z \prec_{\Psi * A} x$, by transitivity, alongside the assumption $y \preccurlyeq_{\Psi * A} x$. We can then apply ($\beta 2_{\preccurlyeq}^*$), using this together with the assumptions $x \in \llbracket A \rrbracket$, $z \in \llbracket \neg A \rrbracket$ and $x \notin \min(\preccurlyeq_\Psi, \llbracket C \rrbracket)$, to obtain the desired result that $z \prec_{\Psi * C} x$. So assume $z \in \llbracket A \rrbracket$. We have now built up the following assumptions about x, y, z: (i) $x \in \llbracket A \wedge C \rrbracket$, (ii) $y \in \llbracket \neg A \wedge C \rrbracket$, and (iii) $z \in \llbracket A \wedge \neg C \rrbracket$. To show the desired result that $z \prec_{\Psi * C} x$ in this final case, it suffices, by ($\gamma 2_{\preccurlyeq}^*$) and (Eq$_{\preccurlyeq}^*$), to show $z \prec_{\Psi * x \vee y} x$, which is equivalent to $\neg (x \vee y) \in [(\Psi * x \vee y) * x \vee z]$ (since $y \neq z \neq x$ from $z \prec_\Psi y \preccurlyeq_\Psi x$). To prove this, it suffices, by ($\Omega 3^*$) and (Eq$_{\preccurlyeq}^*$), to show $\neg (x \vee y) \in [\Psi * x \vee y \vee z]$ and $x \vee z \notin [(\Psi * x \vee z) * x \vee y]$. But the former holds, since we already established $z \prec_\Psi y \preccurlyeq_\Psi x$, while the latter is equivalent to $y \preccurlyeq_{\Psi * x \vee z} x$. This will follow from $y \preccurlyeq_{\Psi * A} x$ and ($\beta 1_{\preccurlyeq}^*$), provided we have $x \notin \min(\preccurlyeq_\Psi, \llbracket x \vee z \rrbracket)$, i.e. $z \prec_\Psi x$. But we have already established that.

(b) **Regarding** (P$_{\preccurlyeq}^*$)**:** We will show that ($\Omega 1^*$) implies (P*), whose equivalence to (P$_{\preccurlyeq}^*$) is well known. So suppose $\neg A \notin [\Psi * B]$. We must show $A \in [(\Psi * A) * B]$. From $\neg A \notin [\Psi * B]$ and the AGM postulates, we obtain $\neg A \notin [\Psi * A \vee B]$ and also $[\Psi * B] \subseteq [(\Psi * B) * A]$. Since $B \in [\Psi * B]$, by Success, the latter gives us $B \in [(\Psi * B) * A]$. Then, from this and $\neg A \notin [\Psi * A \vee B]$, we obtain the required $A \in [(\Psi * A) * B]$ by ($\Omega 1^*$). $\qquad \square$

Proposition 11. (a) *Given AGM, ($\beta 3_{\preccurlyeq}^*$) is equivalent to*

($\beta 3^*$)   *If $B_2 \notin [\Psi * B_1]$, $B_1 \to A \notin [(\Psi * A) * B_2]$, and $B_2 \to \neg A \in [\Psi * C]$, then $B_2 \wedge A \notin [(\Psi * C) * B_1 \veebar B_2]$.*

(b) *Given AGM and (C4$_{\preccurlyeq}^*$), ($\beta 4_{\preccurlyeq}^*$) is equivalent to:*

($\beta 4^*$)   *If $B_2 \notin [\Psi * B_1]$, $B_1 \wedge \neg A \in [(\Psi * A) * B_2]$, and $B_2 \to \neg A \in [\Psi * C]$,*

$\text{then } B_2 \to \neg A \in [(\Psi * C) * B_1 \veebar B_2].$

(c) *Given AGM and* $(C3^*_{\preccurlyeq})$, $(\alpha 3^*_{\preccurlyeq})$ *is equivalent to:*

$(\alpha 3^*)$ *If* $\neg B_2 \in [\Psi * B_1]$, $B_1 \to A \notin [(\Psi * A) * B_2]$, *and* $B_2 \to \neg A \in [\Psi * C]$, *then* $B_2 \to \neg A \in [(\Psi * C) * B_1 \veebar B_2].$

Proof:

(a)   **(i) From** $(\beta 3^*_{\preccurlyeq})$ **to** $(\beta 3^*)$**:** Assume $B_2 \notin [\Psi * B_1]$, $B_1 \to A \notin [(\Psi * A) * B_2]$ and $B_2 \to \neg A \in [\Psi * C]$. From the first assumption, $\exists z \in \min(\preccurlyeq_\Psi, [\![B_1]\!]) \cap [\![\neg B_2]\!]$ and, from the second, $\exists y \in \min(\preccurlyeq_{\Psi * A}, [\![B_2]\!]) \cap [\![B_1 \wedge \neg A]\!]$. From this, we have $y \in [\![B_1]\!]$ and $z \in \min(\preccurlyeq_\Psi, [\![B_1]\!])$, hence: (1) $z \preccurlyeq_\Psi y$.

Assume now for reductio, the negation of the consequent of $(\beta 3^*)$, so that $\min(\preccurlyeq_{\Psi * C}, [\![B_1 \veebar B_2]\!]) \subseteq [\![B_2 \wedge A]\!]$. So we have $\exists x \in \min(\preccurlyeq_{\Psi * C}, [\![B_1 \veebar B_2]\!]) \cap [\![B_2 \wedge A]\!]$. From this, which entails $x \in [\![B_2]\!]$, and the facts that $y \in \min(\preccurlyeq_{\Psi * A}, [\![B_2]\!])$ and $x \in [\![B_2]\!]$, we recover: (2) $y \preccurlyeq_{\Psi * A} x$. From our third initial assumption that $B_2 \to \neg A \in [\Psi * C]$ and the fact that $x \in [\![B_2 \wedge A]\!]$, we obtain: (3) $x \notin \min(\preccurlyeq_\Psi, [\![C]\!])$. Since $z \in [\![\neg B_2]\!]$ and $y \in [\![B_2]\!]$, we also know: (4) $z \neq y$.

As we already know that $x \in [\![A]\!]$ and $y \in [\![\neg A]\!]$, (1), (2), (3) and (4) enable us to apply $(\beta 3^*_{\preccurlyeq})$ to infer $z \preccurlyeq_{\Psi * C} x$. Given that we know that $z \in [\![B_1 \wedge \neg B_2]\!]$ and $x \in \min(\preccurlyeq_{\Psi * C}, [\![B_1 \veebar B_2]\!])$, we can conclude from this last proposition that $z \in \min(\preccurlyeq_{\Psi * C}, [\![B_1 \veebar B_2]\!])$. But we also know that $z \in [\![\neg B_2]\!]$. So we can conclude that $\min(\preccurlyeq_{\Psi * C}, [\![B_1 \veebar B_2]\!]) \nsubseteq [\![B_2 \wedge A]\!]$ after all, as required.

(ii) From $(\beta 3^*)$ **to** $(\beta 3^*_{\preccurlyeq})$**:** Assume the antecedent of $(\beta 3^*_{\preccurlyeq})$: $x \notin \min(\preccurlyeq_\Psi, [\![C]\!])$, $x \in [\![A]\!]$, $y \in [\![\neg A]\!]$, $z \preccurlyeq_\Psi y$, $y \preccurlyeq_{\Psi * A} x$, and $z \neq y$.

If $z = x$, then it follows from this, by reflexivity of $\preccurlyeq_{\Psi * C}$ that $z \preccurlyeq_{\Psi * C} x$ and we are done. So assume henceforth that $z \neq x$.

Assume for reductio that $\neg(x \vee y) \vee \neg A \notin [\Psi * C]$, so that $\exists w \in$ $\min(\preccurlyeq_\Psi, [\![C]\!]) \cap [\![A \wedge (x \vee y)]\!]$. From $x \in [\![A]\!]$ and $y \in [\![\neg A]\!]$, we have $[\![A \wedge (x \vee y)]\!] = \{x\}$. This means that $x \in \min(\preccurlyeq_\Psi, [\![C]\!])$. But we initially assumed this to be false. So we can conclude by reductio:

(1) $\neg(x \vee y) \vee \neg A \in [\Psi * C]$.

Since $z \preccurlyeq_\Psi y$, $z \neq y$ and $z \neq x$: (2) $x \vee y \notin [\Psi * z \vee y]$. Furthermore, from $y \in [\![\neg A]\!]$ and $y \preccurlyeq_{\Psi * A} x$, we can infer: (3) $\neg(z \vee y) \vee A \notin [(\Psi * A) * x \vee y]$.

(1), (2) and (3) then enable us to apply ($\beta 3^*$), with $B_1 = z \vee y$ and $B_2 = x \vee y$, to recover $(x \vee y) \wedge A \notin [(\Psi * C) * (z \vee y) \veebar (x \vee y)]$. Given ($\mathrm{Eq}^*_{\preccurlyeq}$), this allows us to infer $(x \vee y) \wedge A \notin [(\Psi * C) * z \vee x]$, from which it follows, by $x \in [\![A]\!]$, that $z \preccurlyeq_{\Psi * C} x$, as required.

(b) (i) **From ($\beta 4^*_{\preccurlyeq}$) to ($\beta 4^*$):** Assume the antecedent of ($\beta 4^*$): $B_2 \notin [\Psi * B_1]$, $B_1 \wedge \neg A \in [(\Psi * A) * B_2]$ and $B_2 \to \neg A \in [\Psi * C]$. From the first assumption, $\exists z \in \min(\preccurlyeq_\Psi, [\![B_1]\!]) \cap [\![\neg B_2]\!]$ and, from the second, $\min(\preccurlyeq_{\Psi * A}, [\![B_2]\!]) \subseteq [\![B_1 \wedge \neg A]\!]$.

Consider now an arbitrary $y \in \min(\preccurlyeq_{\Psi * A}, [\![B_2]\!])$. By the previous inclusion, we have $y \in [\![B_1]\!]$, and so, since $z \in \min(\preccurlyeq_\Psi, [\![B_1]\!])$: (1) $z \preccurlyeq_\Psi y$.

Assume now for reductio, the negation of the consequent of ($\beta 4^*$), so that $\min(\preccurlyeq_{\Psi * C}, [\![B_1 \veebar B_2]\!]) \nsubseteq [\![B_2 \to \neg A]\!]$. From this, $\exists x \in$ $\min(\preccurlyeq_{\Psi * C}, [\![B_1 \veebar B_2]\!]) \cap [\![B_2 \wedge A]\!]$. It follows from this that $x \in [\![B_2]\!]$ and $x \in [\![A]\!]$ and we already know that $\min(\preccurlyeq_{\Psi * A}, [\![B_2]\!]) \subseteq [\![\neg A]\!]$. Hence: (2) $y \prec_{\Psi * A} x$. From our third initial assumption and the fact that $x \in [\![B_2 \wedge A]\!]$ we can also infer: (3) $x \notin \min(\preccurlyeq_\Psi, [\![C]\!])$. Furthermore, since $z \in [\![\neg B_2]\!]$ and $y \in [\![B_2]\!]$, it follows that (4) $z \neq y$.

As we already know that $x \in [\![A]\!]$ and $y \in [\![\neg A]\!]$, (1), (2), (3) and (4) enable us to apply ($\beta 4^*_{\preccurlyeq}$) to infer $z \prec_{\Psi * C} x$. From this, since $z \in [\![B_1 \wedge \neg B_2]\!]$, it follows that $x \notin \min(\preccurlyeq_{\Psi * C}, [\![B_1 \veebar B_2]\!])$. But this

contradicts our assumption that $x \in \min(\preccurlyeq_{\Psi * C}, [\![B_1 \veebar B_2]\!])$, so we can conclude, by reductio, that $\min(\preccurlyeq_{\Psi * C}, [\![B_1 \veebar B_2]\!]) \subseteq [\![B_2 \to \neg A]\!]$, as required.

(ii) **From** $(\beta 4^*)$ **to** $(\beta 4^*_{\preccurlyeq})$**:** Assume the antecedent of $(\beta 4^*_{\preccurlyeq})$: $x \notin \min(\preccurlyeq_\Psi, [\![C]\!])$, $x \in [\![A]\!]$, $y \in [\![\neg A]\!]$, $z \preccurlyeq_\Psi y$, $y \prec_{\Psi * A} x$ and $z \neq y$

Assume for reductio that $z = x$. Then, by $z \preccurlyeq_\Psi y$, we have $x \preccurlyeq y$. Note that, additionally, we have assumed $y \prec_{\Psi * A} x$. However, $x \in [\![A]\!]$ and $y \in [\![\neg A]\!]$ give us, by $(C4^*_{\preccurlyeq})$: If $x \preccurlyeq_\Psi y$ then $x \preccurlyeq_{\Psi * A} y$. Contradiction. So we can conclude, by reductio, that $z \neq x$.

Assume now for reductio that $\neg(x \vee y) \vee \neg A \notin [\Psi * C]$, so that $\exists w \in \min(\preccurlyeq_\Psi, [\![C]\!]) \cap [\![A \wedge (x \vee y)]\!]$. From $x \in [\![A]\!]$ and $y \in [\![\neg A]\!]$, we already have $[\![A \wedge (x \vee y)]\!] = \{x\}$. So we can infer that $x \in \min(\preccurlyeq_\Psi, [\![C]\!])$, contradicting our initial assumption. So we can conclude, by reductio, that (1) $\neg(x \vee y) \vee \neg A \in [\Psi * C]$, after all.

From $z \preccurlyeq_\Psi y$, $z \neq y$ and $z \neq x$, we recover: (2) $x \vee y \notin [\Psi * z \vee y]$. From $y \in [\![\neg A]\!]$ and $y \prec_{\Psi * A} x$ we can infer: (3) $(z \vee y) \wedge \neg A \in [(\Psi * A) * x \vee y]$.

(1), (2) and (3) then enable us to apply $(\beta 4^*)$, with $B_1 = z \vee y$ and $B_2 = x \vee y$, to recover $(x \vee y) \to \neg A \in [(\Psi * C) * (z \vee y) \veebar (x \vee y)]$. Given $(Eq^*_{\preccurlyeq})$, this allows us to infer $(x \vee y) \to \neg A \in [(\Psi * C) * z \vee x]$, from which it follows, by $x \in [\![A]\!]$, that $z \prec_{\Psi * C} x$, as required.

(c) (i) **From** $(\alpha 3^*_{\preccurlyeq})$ **to** $(\alpha 3^*)$**:** Assume the antecedent of $(\alpha 3^*)$: $\neg B_2 \in [\Psi * B_1]$, $B_1 \to A \notin [(\Psi * A) * B_2]$ and $B_2 \to \neg A \in [\Psi * C]$. From the first principle, we have $\min(\preccurlyeq_\Psi, [\![B_1]\!]) \subseteq [\![\neg B_2]\!]$ and, from the second, $\exists y \in \min(\preccurlyeq_{\Psi * A}, [\![B_2]\!]) \cap [\![B_1 \wedge \neg A]\!]$. Consider now an arbitrary $z \in \min(\preccurlyeq_\Psi, [\![B_1]\!])$. Since $y \in [\![B_1]\!]$ and $y \in [\![B_2]\!]$, it then follows from $\min(\preccurlyeq_\Psi, [\![B_1]\!]) \subseteq [\![\neg B_2]\!]$ that (1) $z \prec_\Psi y$.

Assume now for reductio, the negation of the consequent of $(\alpha 3^*)$, so that $\min(\preccurlyeq_{\Psi * C}, [\![B_1 \veebar B_2]\!]) \nsubseteq [\![B_2 \to \neg A]\!]$. From this, $\exists x \in \min(\preccurlyeq_{\Psi * C}, [\![B_1 \veebar B_2]\!]) \cap [\![B_2 \wedge A]\!]$.

63

From $y \in \min(\preccurlyeq_{\Psi * A}, \llbracket B_2 \rrbracket)$ and $x \in \llbracket B_2 \rrbracket$, we can infer: (2) $y \preccurlyeq_{\Psi * A} x$. From our third initial assumption that $B_2 \to \neg A \in [\Psi * C]$, since $x \in \llbracket B_2 \wedge A \rrbracket$, we can derive: (3) $x \notin \min(\preccurlyeq_\Psi, \llbracket C \rrbracket)$.

As we already know that $x \in \llbracket A \rrbracket$ and $y \in \llbracket \neg A \rrbracket$, (1), (2), and (3) enable us to apply $(\alpha 3^*_{\preccurlyeq})$ to infer $z \prec_{\Psi * C} x$. From this, since $z \in \llbracket B_1 \wedge \neg B_2 \rrbracket$, it follows that $x \notin \min(\preccurlyeq_{\Psi * C}, \llbracket B_1 \veebar B_2 \rrbracket)$. But this contradicts our assumption that $x \in \min(\preccurlyeq_{\Psi * C}, \llbracket B_1 \veebar B_2 \rrbracket)$, so we can conclude, by reductio, that $\min(\preccurlyeq_{\Psi * C}, \llbracket B_1 \veebar B_2 \rrbracket) \subseteq \llbracket B_2 \to \neg A \rrbracket$, as required.

(ii) **From $(\alpha 3^*)$ to $(\alpha 3^*_{\preccurlyeq})$:** Assume the antecedent of $(\alpha 4^*_{\preccurlyeq})$: (a) $x \notin \min(\preccurlyeq_\Psi, \llbracket C \rrbracket)$, (b) $x \in \llbracket A \rrbracket$, (c) $y \in \llbracket \neg A \rrbracket$, (d) $z \prec_\Psi y$, and (e) $y \preccurlyeq_{\Psi * A} x$.

Assume for reductio that $z = x$. Then, by $z \prec_\Psi y$, we have $x \prec_\Psi y$. Note that additionally, we have assumed $y \preccurlyeq_{\Psi * A} x$. However, $x \in \llbracket A \rrbracket$ and $y \in \llbracket \neg A \rrbracket$ give us, by $(C3^*_{\preccurlyeq})$: If $x \prec_\Psi y$ then $x \prec_{\Psi * A} y$. Contradiction. So we can conclude, by reductio, that $z \neq x$.

Assume now for reductio that $\neg(x \vee y) \vee \neg A \notin [\Psi * C]$, so that $\exists w \in \min(\preccurlyeq_\Psi, \llbracket C \rrbracket) \cap \llbracket A \wedge (x \vee y) \rrbracket$. From $x \in \llbracket A \rrbracket$ and $y \in \llbracket \neg A \rrbracket$, we already have $\llbracket A \wedge (x \vee y) \rrbracket = \{x\}$. So we can infer that $x \in \min(\preccurlyeq_\Psi, \llbracket C \rrbracket)$, contradicting our initial assumption. So we can conclude, by reductio, that (1) $\neg(x \vee y) \vee \neg A \in [\Psi * C]$, after all.

From $z \prec_\Psi y$ and $z \neq x$, we recover: (2) $\neg(x \vee y) \in [\Psi * z \vee y]$. From $y \in \llbracket \neg A \rrbracket$ and $y \preccurlyeq_{\Psi * A} x$, we can also infer: (3) $(z \vee y) \to A \notin [(\Psi * A) * x \vee y]$.

(1), (2) and (3) then enable us to apply $(\alpha 3^*)$, with $B_1 = z \vee y$ and $B_2 = x \vee y$, to recover $(x \vee y) \to \neg A \in [(\Psi * C) * (z \vee y) \veebar (x \vee y)]$. Given $(Eq^*_{\preccurlyeq})$, this allows us to infer $(x \vee y) \to \neg A \in [(\Psi * C) * z \vee x]$, from which it follows, by $x \in \llbracket A \rrbracket$, that $z \prec_{\Psi * C} x$, as required. $\square$

Proposition 12. *If $*$ is a POI operator, then:*

    (a) $A \notin [(\Psi * A) * B]$ iff $\min(\lll_\Psi, \llbracket B \rrbracket) \subseteq \llbracket \neg A \rrbracket$ and $\neg B \in [\Psi * A]$, and

    (b) $\neg A \in [(\Psi * A) * B]$ iff $\min(\llll_\Psi, \llbracket B \rrbracket) \subseteq \llbracket \neg A \rrbracket$ and $\neg B \in [\Psi * A]$.

Proof: We simply provide the proof of (a), since the proof of (b) is analogous. The derivation is similar to the one provided for Proposition 8, Part (i), of Booth & Meyer [12]. We first note that $A \notin [(\Psi * A) * B]$ is given semantically by $\min(\preccurlyeq_{\Psi * A}, \llbracket B \rrbracket) \nsubseteq \llbracket A \rrbracket$ and that $\neg B \in [\Psi * A]$ is given by $\min(\preccurlyeq_\Psi, \llbracket A \rrbracket) \subseteq \llbracket \neg B \rrbracket$.

    (i) **Regarding the left-to-right direction:** Assume the truth of the antecedent, i.e. $\min(\preccurlyeq_{\Psi * A}, \llbracket B \rrbracket) \nsubseteq \llbracket A \rrbracket$, so that there exists $y \in \min(\preccurlyeq_{\Psi * A}, \llbracket B \rrbracket) \cap \llbracket \neg A \rrbracket$. Suppose for reductio that the consequent is false, so that either

        (a) $\min(\lll_\Psi, \llbracket B \rrbracket) \nsubseteq \llbracket \neg A \rrbracket$ or

        (b) $\min(\preccurlyeq_\Psi, \llbracket A \rrbracket) \nsubseteq \llbracket \neg B \rrbracket$.

    Assume (a), so that there exists $x \in \min(\lll_\Psi, \llbracket B \rrbracket) \cap \llbracket A \rrbracket$. By $y \in \min(\preccurlyeq_{\Psi * A}, \llbracket B \rrbracket)$ and $x \in \llbracket B \rrbracket$, we have $y \preccurlyeq_{\Psi * A} x$, which, by the definition of POI assignments, amounts to:

        (1) $y \in \min(\preccurlyeq_\Psi, \llbracket A \rrbracket)$, or

        (2) $x, y \notin \min(\preccurlyeq_\Psi, \llbracket A \rrbracket)$ and $r_A(y) \leq_\Psi r_A(x)$

    Since $y \in \llbracket \neg A \rrbracket$, (1) is ruled out, placing us in case (2). Since $x \in \llbracket A \rrbracket$ and $y \in \llbracket \neg A \rrbracket$, $r_A(y) \leq_\Psi r_A(x)$ amounts to $y^- \leq_\Psi x^+$, and hence $y \lll_\Psi x$, contradicting $x \in \min(\lll_\Psi, \llbracket B \rrbracket)$.

    Assume (b), so that there exists $z \in \min(\preccurlyeq_\Psi, \llbracket A \rrbracket) \cap \llbracket B \rrbracket$. Then $\min(\preccurlyeq_{\Psi * A}, \llbracket B \rrbracket) = \min(\preccurlyeq_\Psi, \llbracket A \rrbracket) \cap \llbracket B \rrbracket$. But, since $\min(\preccurlyeq_\Psi, \llbracket A \rrbracket) \cap \llbracket B \rrbracket \subseteq \llbracket A \rrbracket$, we then have $\min(\preccurlyeq_{\Psi * A}, \llbracket B \rrbracket) \subseteq \llbracket A \rrbracket$, directly contradicting out initial assumption.

    (ii) **Regarding the right-to-left direction:** Assume the antecedent, so that:

(a) $\min(\lll_\Psi, [\![B]\!]) \subseteq [\![\neg A]\!]$ and

(b) $\min(\preccurlyeq_\Psi, [\![A]\!]) \subseteq [\![\neg B]\!]$.

We will show that there exists y in $\min(\lll_\Psi, [\![B]\!])$, and hence in $[\![\neg A]\!]$, that is also in $\min(\preccurlyeq_{\Psi * A}, [\![B]\!])$, which will suffice to establish the consequent. Assume then for reductio that for all $y \in \min(\lll_\Psi, [\![B]\!])$, we have $y \notin \min(\preccurlyeq_{\Psi * A}, [\![B]\!])$. So there exists $x \in [\![B]\!]$ such that, for all $y \in \min(\lll_\Psi, [\![B]\!])$, $x \prec_{\Psi * A} y$. Now, by the definition of POI assignments, $x \prec_{\Psi * A} y$ holds iff:

(1) $x \in \min(\preccurlyeq_\Psi, [\![A]\!])$ and $y \notin \min(\preccurlyeq_\Psi, [\![A]\!])$, or

(2) $x, y \notin \min(\preccurlyeq_\Psi, [\![A]\!])$ and $r_A(x) <_\Psi r_A(y)$

Given (b) and $x \in [\![B]\!]$, it follows that $x \notin \min(\preccurlyeq_\Psi, [\![A]\!])$ and so (1) fails. This places us in case (2). Since $y \in \min(\lll, [\![B]\!])$, by (a), we have $y \in [\![\neg A]\!]$ and so $r_A(x) <_\Psi r_A(y)$ gives us $r_A(x) <_\Psi y^-$. From $r_A(x) <_\Psi r_A(y)$, we also have $x \neq y$, and so $x \notin \min(\lll_\Psi, [\![B]\!])$. So, for some $y \in \min(\lll_\Psi, [\![B]\!])$, we must have $y \lll_\Psi x$, that is, $y^- \leq_\Psi x^+$. But this contradicts $r_A(x) <_\Psi y^-$. $\qquad\square$

Proposition 13. *In the presence of AGM,* $(\mathrm{C1}^*_{\preccurlyeq})$ *and* $(\mathrm{C2}^*_{\preccurlyeq})$, **(a)** $(\gamma 1^*_{\preccurlyeq})$ *and* $(\gamma 4^*_{\preccurlyeq})$ *jointly entail* (iDO^*) *and* **(b)** $(\gamma 2^*_{\preccurlyeq})$ *and* $(\gamma 3^*_{\preccurlyeq})$ *jointly entail* (iDR^*).

Proof: We establish the result by deriving the following lemma:

Lemma 4. **(a)** *In the presence of* $(\mathrm{C1}^*_{\preccurlyeq})$ *and* $(\mathrm{C2}^*_{\preccurlyeq})$, $(\gamma 1^*_{\preccurlyeq})$ *and* $(\gamma 4^*_{\preccurlyeq})$ *jointly entail:*

(sWPU*)   *If* $y \preccurlyeq_{\Psi * A} x$ *and* $z \preccurlyeq_{\Psi * C} x$, *then either* $y \preccurlyeq_{\Psi * A \vee C} x$ *or* $z \preccurlyeq_{\Psi * A \vee C} x$

(b) *In the presence of* $(\mathrm{C1}^*_{\preccurlyeq})$ *and* $(\mathrm{C2}^*_{\preccurlyeq})$, $(\gamma 2^*_{\preccurlyeq})$ *and* $(\gamma 3^*_{\preccurlyeq})$ *jointly entail:*

(sSPU*)   *If* $y \prec_{\Psi * A} x$ *and* $z \prec_{\Psi * C} x$, *then either* $y \prec_{\Psi * A \vee C} x$ *or* $z \prec_{\Psi * A \vee C} x$

Given this, the required conclusion follows immediately from Proposition 3 of [37].

(a) We first note that $(C1^*_\preccurlyeq)$ and $(\gamma 1^*_\preccurlyeq)$ jointly entail:

$(s\gamma 1^*_\preccurlyeq)$ If $x \in [\![A]\!]$ and $y \preccurlyeq_{\Psi * A} x$, then $y \preccurlyeq_{\Psi * A \vee C} x$

From this, it follows that $(sWPU^*)$ holds whenever either $x \in [\![A]\!]$ or $x \in [\![C]\!]$. So assume henceforth that $x \in [\![\neg(A \vee C)]\!]$.

Now assume $y \preccurlyeq_{\Psi * A} x$, $z \preccurlyeq_{\Psi * C} x$, and, for contradiction, that both $x \prec_{\Psi * A \vee C} y$ and $x \prec_{\Psi * A \vee C} z$. If either (i) $y \notin \min(\preccurlyeq_\Psi, [\![A]\!])$ and $y \in [\![A]\!]$, or (ii) $y \notin \min(\preccurlyeq_\Psi, [\![A]\!])$ and $y \in [\![C]\!]$ then, by $(\gamma 4^*_\preccurlyeq)$, it follows from $x \in [\![\neg(A \vee C)]\!]$ and $x \prec_{\Psi * A \vee C} y$ that $x \prec_{\Psi * A} y$, contradicting our assumption that $y \preccurlyeq_{\Psi * A} x$. So assume that either $y \in \min(\preccurlyeq_\Psi, [\![A]\!])$ or $y \in [\![\neg(A \vee C)]\!]$. By parallel reasoning from $x \in [\![\neg(A \vee C)]\!]$ and $x \prec_{\Psi * A \vee C} z$, we end up with the assumption that either $z \in \min(\preccurlyeq_\Psi, [\![C]\!])$ or $z \in [\![\neg(A \vee C)]\!]$.

Assume that $y \in [\![\neg(A \vee C)]\!]$. By $(C2^*_\preccurlyeq)$, it then follows from this, $x \in [\![\neg(A \vee C)]\!]$ and $x \prec_{\Psi * A \vee C} y$ that $x \prec_\Psi y$. But from $x \prec_\Psi y$, $x, y \in [\![\neg A]\!]$ and $(C2^*_\preccurlyeq)$ again, we have $x \prec_{\Psi * A} y$, contradicting our assumption that $y \preccurlyeq_{\Psi * A} x$. Similarly, assuming that $z \in [\![\neg(A \vee C)]\!]$ leaves us with $x \prec_{\Psi * C} z$, this time contradicting our assumption that $z \preccurlyeq_{\Psi * C} x$.

So assume that $y \notin [\![\neg(A \vee C)]\!]$ and $z \notin [\![\neg(A \vee C)]\!]$. It follows that both $y \in \min(\preccurlyeq_\Psi, [\![A]\!])$ and $z \in \min(\preccurlyeq_\Psi, [\![C]\!])$.

It follows from $x \prec_{\Psi * A \vee C} y$ that $y \notin \min(\preccurlyeq_\Psi, [\![A \vee C]\!])$. From this and $y \in \min(\preccurlyeq_\Psi, [\![A]\!])$, we obtain $\min(\preccurlyeq_\Psi, [\![A \vee C]\!]) = \min(\preccurlyeq_\Psi, [\![C]\!])$. Since it also follows from $x \prec_{\Psi * A \vee C} z$ that $z \notin \min(\preccurlyeq_\Psi, [\![A \vee C]\!])$, we have $z \notin \min(\preccurlyeq_\Psi, [\![C]\!])$, contradicting our assumption that $z \in \min(\preccurlyeq_\Psi, [\![C]\!])$. Hence either $y \preccurlyeq_{\Psi * A \vee C} x$ or $z \preccurlyeq_{\Psi * A \vee C} x$, as required.

(b) We first note that (C1$^*_{\preccurlyeq}$) and ($\gamma 2^*_{\preccurlyeq}$) jointly entail:

(s$\gamma 2^*_{\preccurlyeq}$) If $x \in [\![A]\!]$ and $y \prec_{\Psi*A} x$, $y \prec_{\Psi*A\vee C} x$

From this, it follows that (sSPU*) holds whenever either $x \in [\![A]\!]$ or $x \in [\![C]\!]$. So assume henceforth that $x \in [\![\neg(A \vee C)]\!]$.

Now assume $y \prec_{\Psi*A} x$, $z \prec_{\Psi*C} x$, and, for contradiction, that both $x \preccurlyeq_{\Psi*A\vee C} y$ and $x \preccurlyeq_{\Psi*A\vee C} z$. If either (i) $y \notin \min(\preccurlyeq_\Psi, [\![A]\!])$ and $y \in [\![A]\!]$, or (ii) $y \notin \min(\preccurlyeq_\Psi, [\![A]\!])$ and $y \in [\![C]\!]$ then, by ($\gamma 3^*_{\preccurlyeq}$), it follows from $x \in [\![\neg(A\vee C)]\!]$ and $x \preccurlyeq_{\Psi*A\vee C} y$ that $x \preccurlyeq_{\Psi*A} y$, contradicting our assumption that $y \prec_{\Psi*A} x$. So assume that either $y \in \min(\preccurlyeq_\Psi, [\![A]\!])$ or $y \in [\![\neg(A \vee C)]\!]$. By parallel reasoning from $x \in [\![\neg(A \vee C)]\!]$ and $x \prec_{\Psi*A\vee C} z$, we end up with the assumption that either $z \in \min(\preccurlyeq_\Psi, [\![C]\!])$ or $z \in [\![\neg(A \vee C)]\!]$.

Assume that $y \in [\![\neg(A \vee C)]\!]$. By (C2$^*_{\preccurlyeq}$), it then follows from this, $x \in [\![\neg(A \vee C)]\!]$ and $x \preccurlyeq_{\Psi*A\vee C} y$ that $x \preccurlyeq_\Psi y$. But from $x \preccurlyeq_\Psi y$, $x, y \in [\![\neg A]\!]$ and (C2$^*_{\preccurlyeq}$) again, we have $x \preccurlyeq_{\Psi*A} y$, contradicting our assumption that $y \prec_{\Psi*A} x$. Similarly, assuming that $z \in [\![\neg(A \vee C)]\!]$ leaves us with $x \preccurlyeq_{\Psi*C} z$, this time contradicting our assumption that $z \prec_{\Psi*C} x$.

So assume that $y \notin [\![\neg(A \vee C)]\!]$ and $z \notin [\![\neg(A \vee C)]\!]$. It follows that both $y \in \min(\preccurlyeq_\Psi, [\![A]\!])$ and $z \in \min(\preccurlyeq_\Psi, [\![C]\!])$.

From the fact that $x \in [\![\neg(A\vee C)]\!]$, it follows from $x \prec_{\Psi*A\vee C} y$ that $y \notin \min(\preccurlyeq_\Psi, [\![A \vee C]\!])$. From this and $y \in \min(\preccurlyeq_\Psi, [\![A]\!])$, we obtain $\min(\preccurlyeq_\Psi, [\![A\vee C]\!]) = \min(\preccurlyeq_\Psi, [\![C]\!])$. Since it also follows, by $x \in [\![\neg(A\vee C)]\!]$, from $x \prec_{\Psi*A\vee C} z$ that $z \notin \min(\preccurlyeq_\Psi, [\![A \vee C]\!])$, we have $z \notin \min(\preccurlyeq_\Psi, [\![C]\!])$, contradicting our assumption that $z \in \min(\preccurlyeq_\Psi, [\![C]\!])$. Hence either $y \prec_{\Psi*A\vee C} x$ or $z \prec_{\Psi*A\vee C} x$, as required. $\qquad\square$

Proposition 14. *Given AGM and* $(\text{C1}^*_{\preccurlyeq})$, **(a)** $(\gamma 1^*_{\preccurlyeq})$ *is equivalent to* (iDI^*) *and* **(b)** $(\gamma 2^*_{\preccurlyeq})$ *is equivalent to* (iK7^*).

Proof: We first establish the following lemma:

Lemma 5. $(\gamma 1^*_{\preccurlyeq})$ *and* $(\gamma 2^*_{\preccurlyeq})$ *are respectively equivalent, in the presence of* $(\text{C1}^*_{\preccurlyeq})$, *to*

$(s\gamma 1^*_{\preccurlyeq})$ *If* $x \in [\![A]\!]$ *and* $y \preccurlyeq_{\Psi * A} x$ *then* $y \preccurlyeq_{\Psi * A \vee C} x$

$(s\gamma 2^*_{\preccurlyeq})$ *If* $x \in [\![A]\!]$ *and* $y \prec_{\Psi * A} x$ *then* $y \prec_{\Psi * A \vee C} x$

$(s\gamma 1^*_{\preccurlyeq})$ and $(s\gamma 2^*_{\preccurlyeq})$ are simply the respective strengthenings of $(\gamma 1^*_{\preccurlyeq})$ and $(\gamma 2^*_{\preccurlyeq})$ in which we do not require $y \in [\![\neg A]\!]$ in the antecedent. So it is sufficient to show that $(\text{C1}^*_{\preccurlyeq})$ entails that, when $x, y \in [\![A]\!]$, the following both hold: (i) if $y \preccurlyeq_{\Psi * A} x$, then $y \preccurlyeq_{\Psi * A \vee C} x$ and (ii) if $y \prec_{\Psi * A} x$, then $y \prec_{\Psi * A \vee C} x$. This is an immediate consequence of the fact that $(\text{C1}^*_{\preccurlyeq})$ entails: If $x, y \in [\![A]\!]$, then $x \preccurlyeq_{\Psi * A} y$ iff $x \preccurlyeq_{\Psi * A \vee C} y$. Regarding the proof of this last implication: It follows from $(\text{C1}^*_{\preccurlyeq})$ that, if $x, y \in [\![A \vee C]\!]$, then $x \prec_{\Psi * A \vee C} y$ iff $x \prec_{\Psi} y$, and hence, since $[\![A]\!] \subseteq [\![A \vee C]\!]$, that:

(1) If $x, y \in [\![A]\!]$, then $x \preccurlyeq_{\Psi * A \vee C} y$ iff $x \preccurlyeq_{\Psi} y$

But $(\text{C1}^*_{\preccurlyeq})$ also directly gives us

(2) If $x, y \in [\![A]\!]$, then $x \preccurlyeq_{\Psi * A} y$ iff $x \preccurlyeq_{\Psi} y$

From (1) and (2), we then recover the required result.

We now return to the proof of the theorem. In view of the above, we now simply need to prove equivalences between (iDI^*) and (iK7^*) and $(s\gamma 1^*_{\preccurlyeq})$ and $(s\gamma 2^*_{\preccurlyeq})$, respectively. Regarding (iK7^*) and $(s\gamma 2^*_{\preccurlyeq})$, it is convenient here to use the following equivalent formulation of (iK7^*):

$$[(\Psi * A) * B] \subseteq \text{Cn}([(\Psi * A \vee C) * B] \cup \{A\})$$

Here, then, is the derivation of the various implications:

(a) (i) **From** $(s\gamma 1^*_{\preccurlyeq})$ **to** (iDI^*): Suppose $\neg A \notin [(\Psi * A \vee C) * B]$. Then there exists $x \in \min(\preccurlyeq_{\Psi*A\vee C}, [\![B]\!]) \cap [\![A]\!]$. Let $y \in \min(\preccurlyeq_{\Psi*A}, [\![B]\!])$. We must show $y \in \min(\preccurlyeq_{\Psi*A\vee C}, [\![B]\!])$. Assume for contradiction that $y \notin \min(\preccurlyeq_{\Psi*A\vee C}, [\![B]\!])$. Then $x \prec_{\Psi*A\vee C} y$. So, since $x \in [\![A]\!]$, by $(s\gamma 1^*_{\preccurlyeq})$, we have $x \prec_{\Psi*A} y$. But this contradicts $y \in \min(\preccurlyeq_{\Psi*A}, [\![B]\!])$. Hence $y \in \min(\preccurlyeq_{\Psi*A\vee C}, [\![B]\!])$, as required.

(ii) **From** (iDI^*) **to** $(s\gamma 1^*_{\preccurlyeq})$: Suppose $x \in [\![A]\!]$ and $x \prec_{\Psi*A\vee C} y$. We must show $x \prec_{\Psi*A} y$. From $x \prec_{\Psi*A\vee C} y$ and $x \in [\![A]\!]$, we know $\neg A \notin [(\Psi * A \vee C) * (x \vee y)]$. So, by (iDI^*), it follows that $[(\Psi * A \vee C) * (x \vee y)] \subseteq [(\Psi * A) * (x \vee y)]$. Moreover $x \prec_{\Psi*A\vee C} y$ gives us $\neg y \in [(\Psi * A \vee C) * (x \vee y)]$, hence $\neg y \in [(\Psi * A) * (x \vee y)]$ and therefore $x \prec_{\Psi*A} y$.

(b) (i) **From** $(s\gamma 2^*_{\preccurlyeq})$ **to** $(iK7^*)$: Establishing $(iK7^*)$ amounts to showing that, if $x \in [\![[(\Psi * A \vee C) * B]]\!] \cap [\![A]\!]$, then $x \in [\![[(\Psi * A) * B]]\!]$. So assume $x \in [\![[(\Psi * A \vee C) * B]]\!] \cap [\![A]\!]$ and, for reductio, $x \notin [\![[(\Psi * A) * B]]\!]$, i.e. there exists $y \in [\![B]\!]$ such that $y \prec_{\Psi*A} x$. Since $x \in [\![A]\!]$, it follows, by $(s\gamma 2^*_{\preccurlyeq})$, that $y \prec_{\Psi*A\vee C} x$ and hence, since $y \in [\![B]\!]$, that $x \notin [\![[(\Psi * A \vee C) * B]]\!]$. Contradiction. Hence $x \in [\![[(\Psi * A) * B]]\!]$, as required.

(ii) **From** $(iK7^*)$ **to** $(s\gamma 2^*_{\preccurlyeq})$: Let $x \in [\![A]\!]$ and $x \preccurlyeq_{\Psi*A\vee C} y$. We must show $x \preccurlyeq_{\Psi*A} y$. From $x \preccurlyeq_{\Psi*A\vee C} y$, we know $x \in \min(\prec_{\Psi*A\vee C}, \{x, y\})$, i.e. $x \in [\![[(\Psi * A \vee C) * (x \vee y)]]\!]$. Since $x \in [\![A]\!]$, it follows by $(iK7^*)$ that $x \in [\![[(\Psi * A) * (x \vee y)]]\!]$, i.e. $x \in \min(\prec_{\Psi*A}, \{x, y\})$. So $x \preccurlyeq_{\Psi*A} y$, as required. $\square$

Proposition 15. $(iK8^*)$ *is equivalent to* $(sP^*_{\preccurlyeq})$, *given AGM and* $(C1^*_{\preccurlyeq})$.

Proof: For convenience, we shall work with the following equivalent formulation of $(iK8^*)$:

If $\neg A \notin [(\Psi * A \vee C) * B]$, then $\mathrm{Cn}([(\Psi * A \vee C) * B] \cup \{A\})$
$\subseteq [(\Psi * A) * B]$

(i) **From** $(\mathrm{sP}^*_{\preccurlyeq})$ **to** $(\mathrm{iK8}^*)$**:** Assume $\neg A \notin [(\Psi * A \vee C) * B]$, i.e. that there exists $x \in \min(\preccurlyeq_{\Psi * A \vee C}, \llbracket B \rrbracket) \cap \llbracket A \rrbracket$. Let $y \in \min(\preccurlyeq_{\Psi * A}, \llbracket B \rrbracket)$. We need to show $y \in \min(\preccurlyeq_{\Psi * A \vee C}, \llbracket B \rrbracket) \cap \llbracket A \rrbracket$. So assume for contradiction that the latter is false, i.e. that one of the following two claims is true: (1) $y \notin \min(\preccurlyeq_{\Psi * A \vee C}, \llbracket B \rrbracket)$ and $y \in \llbracket A \rrbracket$, (2) $y \in \llbracket \neg A \rrbracket$.

Assume (1). Since $x, y \in \llbracket B \rrbracket$, it follows that $x \prec_{\Psi * A \vee C} y$. We have already noted, in the proof of Lemma 5 that the following principle follows from $(\mathrm{C1}^*_{\preccurlyeq})$: If $x, y \in \llbracket A \rrbracket$, then $x \preccurlyeq_{\Psi * A} y$ iff $x \preccurlyeq_{\Psi * A \vee C} y$. Since $x, y \in \llbracket A \rrbracket$ and $x \prec_{\Psi * A \vee C} y$, we therefore have $x \prec_{\Psi * A} y$, contradicting our assumption that $y \in \min(\preccurlyeq_{\Psi * A}, \llbracket B \rrbracket)$.

Assume (2). Since $x \in \llbracket A \rrbracket$ and $x \preccurlyeq_{\Psi * A \vee C} y$, it then follows by $(\mathrm{sP}^*_{\preccurlyeq})$ that $x \prec_{\Psi * A} y$, again contradicting our assumption that $y \in \min(\preccurlyeq_{\Psi * A}, \llbracket B \rrbracket)$.

(ii) **From** $(\mathrm{iK8}^*)$ **to** $(\mathrm{sP}^*_{\preccurlyeq})$**:** Suppose $x \in \llbracket A \rrbracket$, $y \in \llbracket \neg A \rrbracket$ and $x \preccurlyeq_{\Psi * A \vee C} y$. We must show $x \prec_{\Psi * A} y$. From $x \in \llbracket A \rrbracket$ and $x \preccurlyeq_{\Psi * A \vee C} y$, it follows that $\neg A \notin [(\Psi * A \vee C) * x \vee y]$. So by $(\mathrm{iK8}^*)$, we have $\mathrm{Cn}([(\Psi * A \vee C) * x \vee y] \cup \{A\}) \subseteq [(\Psi * A) * x \vee y]$. Since $y \in \llbracket \neg A \rrbracket$, we have $\neg y \in \mathrm{Cn}([(\Psi * A \vee C) * x \vee y] \cup \{A\})$. Therefore $\neg y \in [(\Psi * A) * x \vee y]$ and hence $x \prec_{\Psi * A} y$, as required. $\qquad \square$

Proposition 16. $(\mathrm{sP}^*_{\preccurlyeq})$ *is satisfied by both lexicographic and restrained revision operators.*

Proof: Regarding lexicographic revision, the proof is trivial, since the latter satisfies If $x \in \llbracket A \rrbracket$, $y \in \llbracket \neg A \rrbracket$, then $x \prec_{\Psi * A} y$.

Regarding restrained revision: Assume $x \notin \min(\preccurlyeq_{\Psi}, \llbracket A \rrbracket)$, otherwise the conclusion follows by Success. Assume $y \prec_{\Psi} x$, other wise the conclusion follows

by $(\mathrm{P}^*_{\preccurlyeq})$. From these assumptions, we have $y \prec_{\Psi * A \vee C} x$ by the characteristic principle of restrained revision. $\square$

Proposition 18. *The semantic counterparts of the right-to-left and left-to-right directions of* $(\mathrm{iDF}^*)(\mathrm{i})$ *are respectively:*

$(\gamma 5^*_{\preccurlyeq})$ *If* $x, y \in \llbracket \neg A \rrbracket$ *and* $y \preccurlyeq_{\Psi * A \vee C} x$*, then* $y \preccurlyeq_{\Psi * C} x$

$(\gamma 6^*_{\preccurlyeq})$ *If* $y \in \llbracket \neg A \rrbracket$*, and* $y \prec_{\Psi * A \vee C} x$*, then* $y \prec_{\Psi * C} x$.

Proof: For convenience, we rewrite $(\gamma 5^*_{\preccurlyeq})$ and $(\gamma 6^*_{\preccurlyeq})$ as: If $x, y \in \llbracket \neg C \rrbracket$ and $y \preccurlyeq_{\Psi * A \vee C} x$, then $y \preccurlyeq_{\Psi * A} x$, and, if $y \in \llbracket \neg C \rrbracket$ and $y \prec_{\Psi * A \vee C} x$, then $y \prec_{\Psi * A} x$

  (a) (i) **From** $(\gamma 5^*_{\preccurlyeq})$ **to the right-to-left direction of** $(\mathrm{iDF}^*)(\mathbf{i})$**:** Assume $\min(\preccurlyeq_{\Psi * A \vee C}, \llbracket B \rrbracket) \subseteq \llbracket \neg C \rrbracket$ and $y \in \min(\preccurlyeq_{\Psi * A \vee C}, \llbracket B \rrbracket)$. Assume for reductio that $y \notin \min(\preccurlyeq_{\Psi * A}, \llbracket B \rrbracket)$. Then there exists $x \in \llbracket \neg C \rrbracket$ such that $y \preccurlyeq_{\Psi * A \vee C} x$ but $x \prec_{\Psi * A} y$. Since $y \in \llbracket \neg C \rrbracket$, this contradicts $(\gamma 5^*_{\preccurlyeq})$. So $y \in \min(\preccurlyeq_{\Psi * A}, \llbracket B \rrbracket)$, as required.

        (ii) **From the right-to-left direction of** $(\mathrm{iDF}^*)(\mathbf{i})$ **to** $(\gamma 5^*_{\preccurlyeq})$**:** We consider $(\gamma 5^*_{\preccurlyeq})$ contrapositively. Assume $x, y \in \llbracket \neg C \rrbracket$ and $x \prec_{\Psi * A} y$. From $x, y \in \llbracket \neg C \rrbracket$, it follows that $\neg C \in [(\Psi * A \vee C) * x \vee y]$. From $x \prec_{\Psi * A} y$, we have $x \in [(\Psi * A) * x \vee y]$. Since $\neg C \in [(\Psi * A \vee C) * x \vee y]$, by the right-to-left direction of $(\mathrm{iDF}^*)(\mathrm{i})$, we then have $x \in [(\Psi * A \vee C) * x \vee y]$. It then follows from this that $x \prec_{\Psi * A \vee C} y$, as required.

  (b) (i) **From** $(\gamma 6^*_{\preccurlyeq})$ **to the left-to-right direction of** $(\mathrm{iDF}^*)(\mathrm{i})$**:** Assume $\min(\preccurlyeq_{\Psi * A \vee C}, \llbracket B \rrbracket) \subseteq \llbracket \neg C \rrbracket$ and $x \in \min(\preccurlyeq_{\Psi * A}, \llbracket B \rrbracket)$. Assume for reductio that $x \notin \min(\preccurlyeq_{\Psi * A \vee C}, \llbracket B \rrbracket)$. Then there exists $y \in \llbracket \neg C \rrbracket$ such that $y \prec_{\Psi * A \vee C} x$ but $x \preccurlyeq_{\Psi * A} y$, contradicting $(\gamma 6^*_{\preccurlyeq})$. Hence $x \in \min(\preccurlyeq_{\Psi * A \vee C}, \llbracket B \rrbracket)$, as required.

        (ii) **From the left-to-right direction of** $(\mathrm{iDF}^*)(\mathrm{i})$ **to** $(\gamma 6^*_{\preccurlyeq})$**:** Assume $y \in \llbracket \neg C \rrbracket$ and $y \prec_{\Psi * A \vee C} x$. It follows from this that

72

$\neg C \in [(\Psi * A \vee C) * x \vee y)]$. By the left-to-right direction of (iDF*)(i), we then have $[(\Psi * A \vee C) * x \vee y]] \subseteq [(\Psi * A) * x \vee y]$. Furthermore, $y \prec_{\Psi * A \vee C} x$ leaves us with $\neg x \in [(\Psi * A \vee C) * x \vee y)]$ and hence, by the preceding inclusion $\neg x \in [(\Psi * A) * x \vee y)]$. Hence $y \prec_{\Psi * A} x$, as required. $\square$

Proposition 19. *Neither* $(\gamma 5^*_{\preccurlyeq})$ *nor* $(\gamma 6^*_{\preccurlyeq})$ *are generally satisfied by POI revision operators.*

Proof: Since we know that restrained revision operators are POI revision operators, it will suffice to show that they violate the principle. So consider the countermodel below, where $*$ denotes a restrained revision operator. We have $[\![\neg A \wedge \neg C]\!] \prec_{\Psi * A \vee C} [\![\neg A \wedge C]\!]$, but $[\![\neg A \wedge C]\!] \prec_{\Psi * C} [\![\neg A \wedge \neg C]\!]$, contradicting both principles.

$\square$
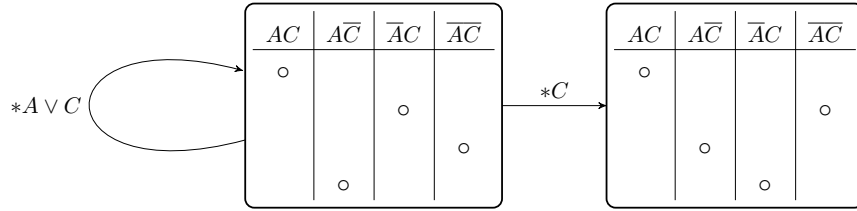
Proposition 20. *Both* $(\gamma 5^*_{\preccurlyeq})$ *and* $(\gamma 6^*_{\preccurlyeq})$ *are satisfied by lexicographic revision operators.*

Proof: We prove the result in relation to $(\gamma 5^*_{\preccurlyeq})$, since the case of $(\gamma 6^*_{\preccurlyeq})$ is analogous. Assume $x, y \in [\![\neg A]\!]$ and $y \preccurlyeq_{\Psi * A \vee C} x$. We consider three cases:

(i) $x \in [\![C]\!]$: Assume for reductio that $y \in [\![\neg C]\!]$. Then, since $x \in [\![A \vee C]\!]$ and $y \in [\![\neg(A \vee C)]\!]$, lexicographic revision yields $x \prec_{\Psi * A \vee C} y$.

Contradiction. Hence $y \in [\![C]\!]$. From $x, y \in [\![A \vee C]\!]$ and $y \preccurlyeq_{\Psi * A \vee C} x$, we obtain $y \preccurlyeq_{\Psi} x$ by $(\mathrm{C1}^*_{\preccurlyeq})$, and from this, in conjunction with $x, y \in [\![C]\!]$ we recover the required result that $y \preccurlyeq_{\Psi * C} x$, again by $(\mathrm{C1}^*_{\preccurlyeq})$.

(ii) $x \in [\![\neg C]\!]$ and $y \in [\![C]\!]$: Then, since $x \in [\![\neg(A \vee C)]\!]$ and $y \in [\![A \vee C]\!]$, lexicographic revision yields $y \preccurlyeq_{\Psi * A \vee C} x$, as required.

(iii) $x, y \in [\![\neg C]\!]$: From $x, y \in [\![\neg(A \vee C)]\!]$ and $y \preccurlyeq_{\Psi * A \vee C} x$, we obtain $y \preccurlyeq_{\Psi} x$ by $(\mathrm{C2}^*_{\preccurlyeq})$, and from this, in conjunction with $x, y \in [\![\neg C]\!]$ we recover the required result that $y \preccurlyeq_{\Psi * C} x$, again by $(\mathrm{C2}^*_{\preccurlyeq})$. $\qquad \square$

References

[1] R. Booth, J. Chandler, On strengthening the logic of iterated belief revision: Proper ordinal interval operators, in: Principles of Knowledge Representation and Reasoning: Proceedings of the Sixteenth International Conference, KR 2018, Tempe, Arizona, 30 October - 2 November 2018., 2018, pp. 210–219.

[2] D. Makinson, P. Gärdenfors, Relations between the logic of theory change and nonmonotonic logic, in: A. Fuhrmann, M. Morreau (Eds.), The Logic of Theory Change, Springer Berlin Heidelberg, Berlin, Heidelberg, 1991, pp. 183–205.

[3] S. O. Hansson, Ten philosophical problems in belief revision, Journal of Logic and Computation 13 (1) (2003) 37–49.

[4] A. Darwiche, J. Pearl, On the logic of iterated belief revision, Artificial Intelligence 89 (1-2) (1997) 1–29.

[5] C. Boutilier, Iterated revision and minimal change of conditional beliefs, Journal of Philosophical Logic 25 (3) (1996) 263–305.

[6] A. C. Nayak, M. Pagnucco, P. Peppas, Dynamic belief revision operators, Artificial Intelligence 146 (2) (2003) 193–228.

[7] R. Booth, T. Meyer, Admissible and restrained revision, Journal of Artificial Intelligence Research 26 (2006) 127–151.

[8] P. Peppas, A panorama of iterated revision, in: David Makinson on classical methods for non-classical problems, Springer, 2014, pp. 71–94.

[9] C. E. Alchourrón, P. Gärdenfors, D. Makinson, On the logic of theory change: Partial meet contraction and revision functions, The Journal of Symbolic Logic 50 (2) (1985) 510–530.

[10] R. Booth, J. Chandler, The irreducibility of iterated to single revision, Journal of Philosophical Logic 46 (4) (2017) 405–418.

[11] Y. Jin, M. Thielscher, Iterated belief revision, revised, Artificial Intelligence 171 (1) (2007) 1–18.

[12] R. Booth, T. Meyer, How to revise a total preorder, Journal of Philosophical Logic 40 (2) (2011) 193–238.

[13] H. Katsuno, A. O. Mendelzon, Propositional knowledge base revision and minimal change, Artificial Intelligence 52 (3) (1991) 263–294.

[14] A. Grove, Two modellings for theory change, Journal of Philosophical Logic 17 (2) (1988) 157–170.

[15] F. P. Ramsey, General propositions and causality (1931).

[16] R. C. Stalnaker, A theory of conditionals, in: N. Rescher (Ed.), Studies in Logical Theory (American Philosophical Quarterly Monographs 2), Oxford: Blackwell, 1968, pp. 98–112.

[17] J. Chandler, Preservation, commutativity and modus ponens: Two recent triviality results, Mind 126 (502) (2017) 579–602.

[18] J. Chandler, Transmission failure, AGM-style, Erkenntnis 78 (2) (2013) 383–398.

[19] J. L. Pollock, Defeasible reasoning, Cognitive Science 11 (4) (1987) 481–518.

[20] K. Schlechta, D. Lehmann, M. Magidor, Distance semantics for belief revision, in: Proceedings of the 6th conference on Theoretical aspects of rationality and knowledge, Morgan Kaufmann Publishers Inc., 1996, pp. 137–145.

[21] P. Peppas, M.-A. Williams, Belief change and semiorders, in: Proceedings of the Fourteenth International Conference on Principles of Knowledge Representation and Reasoning, KR'14, AAAI Press, 2014, pp. 160–167.

[22] M. Pirlot, P. Vincke, Semiorders: Properties, Representations, Applications, Theory and Decision Library B, Springer Netherlands, 2013.

[23] H. Rott, Three floors for the theory of theory change, in: V. P. M. Danák (Ed.), The Logica Yearbook 2013, College Publications, 2014, pp. 187–205.

[24] H. Rott, Four floors for the theory of theory change: The case of imperfect discrimination, in: E. Fermé, J. Leite (Eds.), Logics in Artificial Intelligence, Springer International Publishing, Cham, 2014, pp. 368–382.

[25] S. Konieczny, R. P. Pérez, Improvement operators, in: G. Brewka, J. Lang (Eds.), Principles of Knowledge Representation and Reasoning: Proceedings of the Eleventh International Conference, KR 2008, Sydney, Australia, September 16-19, 2008, AAAI Press, 2008, pp. 177–187.

[26] S. Konieczny, M. M. Grespan, R. P. Pérez, Taxonomy of improvement operators and the problem of minimal change, in: F. Lin, U. Sattler, M. Truszczynski (Eds.), Principles of Knowledge Representation and Reasoning: Proceedings of the Twelfth International Conference, KR 2010, Toronto, Ontario, Canada, May 9-13, 2010, AAAI Press, 2010.

[27] O. Papini, Iterated revision operations stemming from the history of an agent's observations, in: Frontiers in Belief Revision, Springer Netherlands, Dordrecht, 2001, pp. 279–301.

[28] I. Levi, Subjunctives, dispositions and chances, Synthèse 34 (4) (1977) 423–455.

[29] P. Gärdenfors, Knowledge in flux : modeling the dynamics of epistemic states, MIT Press Cambridge, Mass, 1988.

[30] W. Spohn, Ordinal conditional functions. a dynamic theory of epistemic states, in: W. L. Harper, B. Skyrms (Eds.), Causation in Decision, Belief Change, and Statistics, vol. II, Kluwer Academic Publishers, 1988.

[31] W. Spohn, The Laws of Belief: Ranking Theory and its Philosophical Applications, Oxford University Press, 2012.

[32] S. Konieczny, R. P. Pérez, A framework for iterated revision, Journal of Applied Non-Classical Logics 10 (3-4) (2000) 339–367.

[33] D. Lehmann, Belief revision, revised, in: Proceedings of the 14th International Joint Conference on Artificial Intelligence - Volume 2, IJCAI'95, Morgan Kaufmann Publishers Inc., San Francisco, CA, USA, 1995, pp. 1534–1540.

[34] J. P. Delgrande, D. Dubois, J. Lang, Iterated revision as prioritized merging, in: Proceedings, Tenth International Conference on Principles of Knowledge Representation and Reasoning, Lake District of the United Kingdom, June 2-5, 2006, 2006, pp. 210–220.

[35] B. Nebel, How Hard is it to Revise a Belief Base?, Springer Netherlands, Dordrecht, 1998, pp. 77–145.

[36] P. Liberatore, The complexity of iterated belief revision, in: F. Afrati, P. Kolaitis (Eds.), Database Theory — ICDT '97, Springer Berlin Heidelberg, Berlin, Heidelberg, 1997, pp. 276–290.

[37] R. Booth, J. Chandler, Extending the Harper Identity to iterated belief change, in: Proceedings of the Twenty-Fifth International Joint Conference on Artificial Intelligence, IJCAI 2016, New York, NY, USA, 9-15 July 2016, 2016, pp. 987–993.